Two Stage Decumulation Strategies for DC Plan Investors

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Abstract

Optimal stochastic control methods are used to examine decumulation strategies for a defined contribution (DC) plan retiree. An initial investment horizon of fifteen years is considered, since the retiree will attain this age with high probability. The objective function reward measure is the expected sum of the withdrawals. The objective function tail risk measure is the expected linear shortfall with respect to a desired lower bound for wealth at fifteen years. The lower bound wealth level is the amount which is required to fund a lifelong annuity fifteen years after retirement, which generates the required minimum cash flows. This ameliorates longevity risk. The controls are the withdrawal amount each year, and the asset allocation strategy. Maximum and minimum withdrawal amounts are specified. Specifying a short initial decumulation horizon, results in the optimal strategy achieving: (i) median withdrawals at the maximum rate within 2-3 years of retirement (ii) terminal wealth larger than the desired lower bound at fifteen years, with greater than 90% probability and (iii) median terminal wealth at fifteen years considerably larger than the desired lower bound. The controls are computed using a parametric model of historical stock and bond returns, and then tested in bootstrap resampled simulations using historical data. At the fifteen year investment horizon, the retiree has the option of (i) continuing to self-manage the decumulation policy or (ii) purchasing an annuity.

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1 Introduction

It is clear that there is an international trend towards deprecation of defined benefit (DB) pension plans in favour of defined contribution (DC) plans.[1] This is simply because corporations and governments are not prepared to take on the risk of DB plans.

The holder of a DC plan has two challenges. The first challenge is to devise an investment strategy which will accumulate significant wealth by the time of retirement. The second challenge is managing the decumulation strategy during retirement. This paper focuses on the decumulation phase of a DC plan.

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[1]See, for example, *"The extinction of defined-benefit plans is almost upon us,"* Globe and Mail, October 4, 2018. `https://www.theglobeandmail.com/investing/personal-finance/retirement/article-the-extinction-of-defined-benefit-pension-plans-is-almost-upon-us/`

Following the maxim *"a goal properly set is halfway reached,"* [2] our objective in this paper is to set an appropriate goal for a decumulation strategy. Once the goal is set, then this generates an investment/withdrawal strategy for the decumulation phase.

There is a stream of academic literature that suggests that DC plan holders should purchase annuities upon retirement. However, in practice, this is very unpopular with DC plan holders (MacDonald et al., 2013; Peijnenburg et al., 2016). In fact MacDonald et al. (2013) list many reasons for the lack of interest in annuities, including meager payouts in the current low interest rate environment, poor pricing due to adverse selection, no true inflation protection, counterparty risk, and no liquidity.

Another stream of literature focuses on the optimal timing of annuity purchase (Milevsky, 1998; Gerrard et al., 2006; Milevsky and Young, 2007; Di-Giacinto and Vigna, 2012). Essentially, this research is based on the idea that a DC plan retiree may be better off investing in a mix of stocks and bonds, until she reaches an age where the *mortality credits* from an annuity give better returns than the stock-bond portfolio.

Although many suggestions for decumulation strategies are based on maximizing traditional utility functions (see e.g. Bernhardt and Donnelly (2018)), practitioners advocate decumulation strategies which provide minimum (real) cash flows each year to fund expenses (Tretiakova and Yamada, 2011).

This has led to the popularity of various heuristic rules of thumb for asset allocation and decumulation strategies (Bernhardt and Donnelly, 2018). An example is the ubiquitous *four per cent* rule. Based on historical backtests, Bengen (1994) suggests investing in a portfolio of 50% bonds and 50% stocks, and withdrawing 4% of the initial capital each year (adjusted for inflation). Over historical rolling year 30 year periods, this strategy would have never depleted the portfolio.

Recently, the decumulation/investment problem has been posed as a problem in optimal stochastic control (Forsyth, 2021). Realistic constraints on the controls were applied, including minimum and maximum withdrawal amounts per year, and no-leverage, no-shorting constraints on the asset allocation policy. This required numerical solution of a Hamilton-Jacobi-Bellman equation. Similarly to the four per cent rule heuristic, longevity risk was not explicitly taken into account.

The objective function in Forsyth (2021) involved minimizing tail risk (essentially CVAR) and maximizing the total withdrawals, over a fixed thirty year period, assuming the DC plan holder retires at age 65. Choosing the fixed thirty period was regarded as a conservative, practical choice of investment horizon. Of course, one could simply specify the longest possible lifespan for a 65 year old, but this would result in very low withdrawal amounts. Even the thirty year horizon might be construed as overly conservative, since a 65 year old Canadian male has only a 0.13 chance of reaching age 95 [3]. The optimal control for withdrawals in Forsyth (2021) can be summarized as follows: the median optimal strategy is to withdraw at the minimum rate during the early stages of retirement (i.e. 5-10 years), and then to withdraw at the maximum rate during the later years.

It is easy to understand why this strategy is optimal in terms of maximizing expected withdrawals, and minimizing tail risk. The investor withdraws the minimum amount during the early stage of retirement, which avoids the possibility of large withdrawals during market downturns, hence reducing sequence of return risk. Then, with a smaller time remaining, and a high probability of larger wealth accumulation, it is safe to withdraw at the maximum rate. However, this strategy is somewhat disappointing. Although clearly optimal, in the sense of maximizing the objective function, most retirees would prefer to withdraw larger amounts in the early stages of retirement, when they are more active.

[2] Widely attributed to Abraham Lincoln, although this may be apocryphal.

[3] Canadian pensioners mortality, CPM2014, Canadian Society of Actuaries, `www.cia-ica.ca/docs/default-source/2014/214013e.pdf`

Consequently, in this paper, we modify the approach in Forsyth (2021). Our first modification is to use a smaller investment horizon (fifteen years in this case). Secondly, our objective function is to maximize total withdrawals, and to minimize risk as measured by a linear shortfall with respect to a *fixed* target minimum wealth at 15 years. This approach has the following advantages

(i) The shorter time horizon forces larger withdrawals earlier.

(ii) The probability that a Canadian male, retiring at age 65, attains the age of 80 is about 0.76. Hence living to this age (15 years after retirement) is a high probability event.

(iii) We use a fixed shortfall target wealth, in contrast to a CVAR-type risk measure. Our target wealth is the amount which would be required to fund a lifetime annuity (starting at the age of 80), which generates the minimum required cash flows. This ameliorates longevity risk.

(iv) Use of a fixed shortfall target is inherently time consistent and is an intuitive, easily explained risk measure. CVAR-type risk measures are a common tail risk measure in finance. However, CVAR risk measures result in strategies which are not formally time consistent, but are implementable, since a CVAR-type risk measure generates an induced time consistent strategy (Forsyth, 2020a). Fixed target shortfall risk avoids this sort of complication.

As noted above, retirees are reluctant to purchase annuities. Our strategy attempts to maximize withdrawals earlier, while still maintaining high median wealth values after 15 years. At the end of the 15 year investment horizon, the retiree can then embark on the second stage of the decumulation process. In the event that the investments have performed well, the retiree can continue self-managing the decumulation process. However, assuming poor investment results, as characterized by the mean of the worst 5% of outcomes, the investor still has enough wealth to purchase a lifetime annuity, satisfying minimum required cash flows. A risk averse investor might choose to annuitize at this point, taking advantage of mortality credits, and the longevity hedge of annuities.

Our focus in this paper is on the first stage (to 15 years) of the decumulation process. At the end of 15 years, the retiree has to decide on the tradeoff between maximizing withdrawals, hedging longevity risk, and managing the portfolio. At this point, it is probably not possible to propose any sort of general strategy, since the choices amongst the different policies will depend crucially on each individual's preferences. Consequently, our suggested two stage decumulation strategy generates useful advice for most retirees for the initial decumulation period (15 years), leaving the strategy for the second stage to be determined on a case by case basis.

We determine the parameters of our stochastic process model for stock and bond indices by calibration to historical data in the range 1926:1-2019:12. All stochastic process models are real (inflation adjusted). The optimal strategies are computed based on the calibrated parametric stochastic processes. We refer to simulated market based on the calibrated parametric stochastic processes as the *synthetic* market. We test for robustness by applying the controls computed in the synthetic market to simulations based on stationary block bootstrapped historical data (Politis and Romano, 1994; Politis and White, 2004; Patton et al., 2009; Dichtl et al., 2016). We refer to the market based on bootstrapping historical data as the *historical* market.

Our main results are

(i) Compared to a minimum risk strategy, median withdrawals in the early years of retirement can be increased significantly, with very small increases in tail risk. This is a direct result of allowing some flexibility in the withdrawals.

(ii) The strategy is robust, in the sense that the efficient frontiers, computed using the controls determined in the synthetic market, are virtually identical in the historical and synthetic markets.

This would appear to suggest that a good strategy for DC plan decumulation is to break up the investment horizon into early and late periods. This allows more flexibility in terms of strategies, with no need to pre-commit for very long periods. We suggest that our strategy is applicable to many retirees during the early stage of retirement. During the later stage of retirement, we agree with Bernhardt and Donnelly (2018), that (referring to decumulation strategies) *"There is no solution that is appropriate for everyone and neither is there a single solution for any individual."*

2 Formulation

We assume that the investor has access to two funds: a broad market stock index fund and a constant maturity bond index fund.

The investment horizon is T. Let S_t and B_t respectively denote the real (inflation adjusted) *amounts* invested in the stock index and the bond index respectively. In the absence of an investor determined control (i.e. cash withdrawals or rebalancing), all changes in S_t and B_t result from changes in asset prices. We model the stock index as following a jump diffusion.

In addition, we follow the usual practitioner approach and directly model the returns of the constant maturity bond index as a stochastic process, see for example Lin et al. (2015); MacMinn et al. (2014).

Let $S_{t^-} = S(t-\epsilon), \epsilon \to 0^+$, i.e. t^- is the instant of time before t, and let ξ^s be a random number representing a jump multiplier. When a jump occurs, $S_t = \xi^s S_{t^-}$. Allowing for jumps permits modelling of non-normal asset returns. We assume that $\log(\xi^s)$ follows a double exponential distribution (Kou, 2002; Kou and Wang, 2004). If a jump occurs, p_u^s is the probability of an upward jump, while $1 - p_u^s$ is the chance of a downward jump. The density function for $y = \log(\xi^s)$ is

$$f^s(y) = p_u^s \eta_1^s e^{-\eta_1^s y} \mathbf{1}_{y \geq 0} + (1 - p_u^s) \eta_2^s e^{\eta_2^s y} \mathbf{1}_{y < 0} \ , \tag{2.1}$$

where η_1^s (η_2^s) is the exponential distribution parameter for an up (down) stock jump. We also define

$$\kappa_\xi^s \quad = \quad E[\xi^s - 1] = \frac{p_u^s \eta_1^s}{\eta_1^s - 1} + \frac{(1 - p_u^s)\eta_2^s}{\eta_2^s + 1} - 1 \ . \tag{2.2}$$

In the absence of control, S_t evolves according to

$$\frac{dS_t}{S_{t^-}} \quad = \quad \left(\mu^s - \lambda_\xi^s \kappa_\xi^s\right) \, dt + \sigma^s \, dZ^s + d\left(\sum_{i=1}^{\pi_t^s} (\xi_i^s - 1)\right) , \tag{2.3}$$

where μ^s is the (uncompensated) drift rate, σ^s is the volatility, dZ^s is the increment of a Wiener process, π_t^s is a Poisson process with positive intensity parameter λ_ξ^s, and ξ_i^s are i.i.d. positive random variables having distribution (2.1). Moreover, ξ_i^s, π_t^s, and Z^s are assumed to all be mutually independent.

Similarly, let the amount in the bond index be $B_{t^-} = B(t-\epsilon), \epsilon \to 0^+$. As in MacMinn et al. (2014), we assume that the constant maturity bond index follows a jump diffusion process. Consequently, in the absence of control, B_t evolves as

$$\frac{dB_t}{B_{t^-}} \quad = \quad \left(\mu^b - \lambda_\xi^b \kappa_\xi^b + \mu_c^b \mathbf{1}_{\{B_{t^-} < 0\}}\right) \, dt + \sigma^b \, dZ^b + d\left(\sum_{i=1}^{\pi_t^b} (\xi_i^b - 1)\right) , \tag{2.4}$$

where the terms in equation (2.4) are defined analogously to equation (2.3). In particular, π_t^b is a Poisson process with positive intensity parameter λ_ξ^b, and ξ_i^b has distribution

$$f^b(y = \log \xi^b) = p_u^b \eta_1^b e^{-\eta_1^b y} \mathbf{1}_{y \geq 0} + (1 - p_u^b) \eta_2^b e^{\eta_2^b y} \mathbf{1}_{y<0} \ , \tag{2.5}$$

where where η_1^b (η_2^b) is the exponential distribution parameter for an up (down) bond jump. and $\kappa_\xi^b = E[\xi^b - 1]$. ξ_i^b, π_t^b, and Z^b are assumed to all be mutually independent. The term $\mu_c^b \mathbf{1}_{\{B_{t^-}<0\}}$ in equation (2.4) represents the extra cost of borrowing (the spread).

The diffusion processes are correlated, i.e. $dZ^s \cdot dZ^b = \rho_{sb} dt$. The stock and bond jump processes are assumed mutually independent. See Forsyth (2020b) for justification of the assumption of stock-bond jump independence.

We define the investor's total wealth at time t as

$$\text{Total wealth } \equiv W_t = S_t + B_t. \tag{2.6}$$

We impose the constraints that (assuming solvency) shorting stock and using leverage (i.e. borrowing) are not permitted, which would be typical of a DC plan retirement savings account. In the event of insolvency (due to withdrawals), the portfolio is liquidated, trading ceases and debt accumulates at the borrowing rate. In the insolvency case, it is assumed that the investor continues to withdraw (i.e. borrow) from the account.

3 Notational conventions

Consider a set of discrete withdrawal/rebalancing times $\mathcal{T}$

$$\mathcal{T} = \{t_0 = 0 < t_1 < t_2 < \ldots < t_M = T\} \tag{3.1}$$

where we assume that $t_i - t_{i-1} = \Delta t = T/M$ is constant for simplicity. To avoid subscript clutter, in the following, we will occasionally use the notation $S_t \equiv S(t), B_t \equiv B(t)$ and $W_t \equiv W(t)$. Let the inception time of the investment be $t_0 = 0$. We let $\mathcal{T}$ be the set of withdrawal/rebalancing times, as defined in equation (3.1). At each rebalancing time t_i, $i = 0, 1, \ldots, M-1$, the investor (i) withdraws an amount of cash q_i from the portfolio, and then (ii) rebalances the portfolio. At $t_M = T$, the final cash flow q_M occurs, and the portfolio is liquidated. In the following, given a time dependent function $f(t)$, then we will use the shorthand notation

$$f(t_i^+) \equiv \lim_{\epsilon \to 0^+} f(t_i + \epsilon) \quad ; \quad f(t_i^-) \equiv \lim_{\epsilon \to 0^+} f(t_i - \epsilon) \ . \tag{3.2}$$

We assume that there are no taxes or other transaction costs, so that the condition

$$W(t_i^+) = W(t_i^-) - q_i \quad ; \quad t_i \in \mathcal{T} \tag{3.3}$$

holds. Typically, DC plan savings are held in a tax advantaged account, with no taxes triggered by rebalancing. With infrequent (e.g. yearly) rebalancing, we also expect transaction costs to be small, and hence can be ignored. It is possible to include transaction costs, but at the expense of increased computational cost (Staden et al., 2018).

We denote by $X(t) = (S(t), B(t))$, $t \in [0,T]$, the multi-dimensional controlled underlying process, and by $x = (s, b)$ the realized state of the system. Let the rebalancing control $p_i(\cdot)$ be the fraction invested in the stock index at the rebalancing date t_i, i.e.

$$p_i\left(X(t_i^-)\right) = p\left(X(t_i^-), t_i\right) \quad = \quad \frac{S(t_i^+)}{S(t_i^+) + B(t_i^+)} \ . \tag{3.4}$$

Let the withdrawal control $q_i(\cdot)$ be the amount withdrawn at time t_i, i.e. $q_i\left(X(t_i^-)\right) = q\left(X(t_i^-), t_i\right)$. Note that formally, the controls depend on the state of the investment portfolio, before the rebalancing occurs, i.e. $p_i(\cdot) = p\left(X(t_i^-), t_i)\right) = p\left(X_i^-, t_i\right)$, and $q_i(\cdot) = q\left(X(t_i^-), t_i)\right) = q\left(X_i^-, t_i\right)$, $t_i \in \mathcal{T}$, where $\mathcal{T}$ is the set of rebalancing times.

However, it will be convenient to note that in our case, we find the optimal control $p_i(\cdot)$ amongst all strategies with constant wealth (after withdrawal of cash). Hence, with some abuse of notation, we will now consider $p_i(\cdot)$ to be function of wealth after withdrawal of cash

$$\begin{aligned} p_i(\cdot) \quad &= \quad p(W(t_i^+), t_i) \\ & \quad\;\;\; W(t_i^+) = S(t_i^-) + B(t_i^-) - q_i(\cdot) \\ & \quad\;\;\; S(t_i^+) = S_i^+ = p_i(W_i^+)\ W_i^+ \quad ; \quad B(t_i^+) = B_i^+ = (1 - p_i(W_i^+))\ W_i^+ \quad . \end{aligned} \tag{3.5}$$

A control at time t_i, is then given by the pair $(q_i(\cdot), p_i(\cdot))$ where the notation $(\cdot)$ denotes that the control is a function of the state.

Let $\mathcal{Z}$ represent the set of admissible values of the controls $(q_i(\cdot), p_i(\cdot))$. As is typical for a DC plan savings account, we impose no-shorting, no-leverage constraints (assuming solvency). We also impose maximum and minimum values for the withdrawals. We apply the constraint that in the event of insolvency due to withdrawals ($W(t_i^+) < 0$), trading ceases and debt (negative wealth) accumulates at the appropriate bond rate of return (including a spread). We also specify that the stock assets are liquidated at $t = t_M$.

More precisely, let W_i^+ be the wealth after withdrawal of cash, then define

$$\mathcal{Z}_q \quad = \quad [q_{\min}, q_{\max}] \ , \tag{3.6}$$

$$\mathcal{Z}_p(W_i^+, t_i) \quad = \quad \begin{cases} [0,1] & W_i^+ > 0 \ ; \ t_i \in \mathcal{T} \ ; \ t_i \neq t_M \\ \{0\} & W_i^+ \leq 0 \ ; \ t_i \in \mathcal{T} \ ; \ t_i \neq t_M \\ \{0\} & t_i = t_M \end{cases} \ . \tag{3.7}$$

(3.8)

The set of admissible values for $(q_i, p_i), t_i \in \mathcal{T}$, can then be written as

$$(q_i, p_i) \in \mathcal{Z}(W_i^+, t_i) \quad = \quad \mathcal{Z}_q \times \mathcal{Z}_p(W_i^+, t_i) \ . \tag{3.9}$$

For implementation purposes, we have written equation (3.9) in terms of the wealth after withdrawal of cash. However, we remind the reader that since $W_i^+ = W_i^- - q$, the controls are formally a function of the state $X(t_i^-)$ before the control is applied.

The admissible control set $\mathcal{A}$ can then be written as

$$\mathcal{A} = \left\{ (q_i, p_i)_{0 \leq i \leq M} : (q_i, p_i) \in \mathcal{Z}(W_i^+, t_i) \right\} \tag{3.10}$$

An admissible control $\mathcal{P} \in \mathcal{A}$, where $\mathcal{A}$ is the admissible control set, can be written as,

$$\mathcal{P} = \{ (q_i(\cdot), p_i(\cdot)) \ : \ i = 0, \ldots, M \} \ . \tag{3.11}$$

We also define $\mathcal{P}_n \equiv \mathcal{P}_{t_n} \subset \mathcal{P}$ as the tail of the set of controls in $[t_n, t_{n+1}, \ldots, t_M]$, i.e.

$$\mathcal{P}_n = \{ (q_n(\cdot), p_n(\cdot)) \ldots, (p_M(\cdot), q_M(\cdot)) \} \ . \tag{3.12}$$

For notational completeness, we also define the tail of the admissible control set $\mathcal{A}_n$ as

$$\mathcal{A}_n = \left\{ (q_i, p_i)_{n \leq i \leq M} : (q_i, p_i) \in \mathcal{Z}(W_i^+, t_i) \right\} \tag{3.13}$$

so that $\mathcal{P}_n \in \mathcal{A}_n$.

4 Risk and reward

4.1 A measure of risk: definition of linear shortfall (LS)

Let $E[\cdot]$ be the expectation operator, and, given a shortfall target W^*, we define the linear shortfall w.r.t. W^*, LS_{W^*}

$$\text{LS}_{W^*} \;\; = \;\; E[\min(W_T - W^*,0)] \ , \tag{4.1}$$

where $W_T = W(T)$, i.e. the terminal wealth. Note that since we have used W_T in equation (4.1) (final wealth, not loss), our objective is to *maximize* LS_{W^*}.

This risk measure is closely related to Conditional Value at Risk (CVAR). To see this, given an expectation operator $E[\cdot]$, as noted by Rockafellar and Uryasev (2000), CVAR(α) can be written as

$$\text{CVAR}_\alpha \;\; = \;\; \sup_{W^*} E\left[W^* + \frac{1}{\alpha}\min(W_T - W^*, 0)\right] \ . \tag{4.2}$$

CVAR(α) has the convenient interpretation as the mean of the worst α fraction of outcomes. Typically $\alpha \in \{.01, .05\}$. Note that the definition of CVAR in equation (4.2) uses the probability density of the final wealth distribution, not the density of *loss*. Hence, a larger value of CVAR (i.e. a larger value of average worst case terminal wealth) is desired.

CVAR is not formally time consistent (Forsyth, 2020a). However, if we maximize CVAR at time zero, (for a given value of initial wealth) which effectively specifies the VAR value W^*, and then recompute the optimal control at future times (with this fixed value of W^*), then this strategy is an induced time consistent strategy, and hence is implementable (Strub et al., 2017; 2019; Forsyth, 2020a). However, in our context, it is more natural to specify W^* at time zero. W^* is the (real) estimate of the cost of an annuity purchased at the terminal time T, which (possibly combined with other assets) would generate the required minimum cash flows.

Since W^* is fixed, then this risk measure will trivially generate a time consistent control. From the definitions (4.1) and (4.2), we have the following result. Suppose

$$E[\mathbf{1}_{W_T < W^*}] \;\; = \;\; \alpha \ , \tag{4.3}$$

then the relationship between CVAR_α and LS_{W^*} is

$$\text{CVAR}_\alpha \;\; = \;\; W^* + \frac{\text{LS}_{W^*}}{\alpha} \ . \tag{4.4}$$

Remark 4.1 (CVAR LS relationship)**.** *In general, for arbitrary W^*, but fixed α, equation (4.4) will not be valid.*

4.2 A measure of reward: expected total withdrawals (EW)

We will use expected total withdrawals as a measure of reward in the following. More precisely, we define EW (expected total withdrawals) as

$$EW = E\left[\sum_{i=0}^{i=M} q_i\right] \ . \tag{4.5}$$

5 Objective Function

Define $X_0^+ = X(t_0^+), X_0^- = X(t_0^-)$. Since expected withdrawals (EW) and linear shortfall (LS) are conflicting measures, we use a scalarization technique to find the Pareto optimal points for this multi-objective optimization problem. Informally, for a given scalarization parameter $\kappa > 0$, we seek to find the control $\mathcal{P}_0$ that maximizes

$$\text{EW}(X_0^-, t_0^-) + \kappa \ \text{LS}_{W^*}(X_0^-, t_0^-) \quad = \quad E_{\mathcal{P}_0}^{X_0^+, t_0^+}\left[\sum_{i=0}^{i=M} q_i + \kappa \Big(\min(W_T - W^*, 0) \Big) \right] . \tag{5.1}$$

More precisely, we define the EW-LS problem $EWLS_{t_0}(\kappa)$ in terms of the value function $V(s,b,t_0^-)$, with $(s,b,t) \in \Omega = [0,\infty) \times (-\infty, +\infty) \times [0, \infty)$.

$$(EWLS_{t_0}(\kappa)) : \qquad V\left(s,b,t_0^-\right) = \max_{\mathcal{P}_0 \in \mathcal{A}} \Bigg\{ E_{\mathcal{P}_0}^{X_0^+, t_0^+} \Bigg[\sum_{i=0}^{i=M} q_i \ + \ \kappa \Big(\min(W_T - W^*, 0) \Big) \Bigg| X(t_0^-) = (s,b) \Bigg] \Bigg\} \tag{5.2}$$

$$\text{subject to} \qquad \begin{cases} (S_t, B_t) \text{ follow processes (2.3) and (2.4);} \quad t \notin \mathcal{T} \\ W_\ell^+ = S_\ell^- + B_\ell^- - q_\ell \,; \ X_\ell^+ = (S_\ell^+, B_\ell^+) \\ S_\ell^+ = p_\ell(\cdot) W_\ell^+ \,; \ B_\ell^+ = (1 - p_\ell(\cdot)) W_\ell^+ \\ (q_\ell(\cdot), p_\ell(\cdot)) \in \mathcal{Z}(W_\ell^+, t_\ell) \\ \ell = 0, \ldots, M \ ; \ t_\ell \in \mathcal{T} \end{cases} \tag{5.3}$$

6 Dynamic programming solution of the EW-LS problem

We use standard dynamic programming methods to solve problem (5.2). We define the value function at time t_n^-,

$$V(s,b,t_n^-) \quad = \quad \max_{\mathcal{P}_n \in \mathcal{A}_n} \left\{ E_{\mathcal{P}_n}^{\hat{X}_n^+, t_n^+} \left[\sum_{i=n}^{i=M} q_i + \kappa \Big(\min((W_T - W^*), 0) \Big) \Big| \hat{X}(t_n^-) = (s,b) \right] \right\} . \tag{6.1}$$

$$\text{subject to} \qquad \begin{cases} (S_t, B_t) \text{ follow processes (2.3) and (2.4);} \quad t \notin \mathcal{T} \\ W_\ell^+ = S_\ell^- + B_\ell^- - q_\ell \,; \ X_\ell^+ = (S_\ell^+, B_\ell^+) \\ S_\ell^+ = p_\ell(\cdot) W_\ell^+ \,; \ B_\ell^+ = (1 - p_\ell(\cdot)) W_\ell^+ \\ (q_\ell(\cdot), p_\ell(\cdot)) \in \mathcal{Z}(W_\ell^+, t_\ell) \\ \ell = n, \ldots, M \ ; \ t_\ell \in \mathcal{T} \end{cases} . \tag{6.2}$$

Recalling the definitions of $\mathcal{Z}_p, \mathcal{Z}_q$ in equations (3.6-3.7), then the dynamic programming principle applied at $t_n \in \mathcal{T}$ would then imply

$$\begin{aligned} V(s,b,t_n^-) &= \max_{q \in \mathcal{Z}_q} \quad \max_{p \in \mathcal{Z}_p(w^- - q, t)} \left\{ q + \left[V((w^- - q)p, (w^- - q)(1-p), t_n^+) \right] \right\} \\ &= \max_{q \in \mathcal{Z}_q} \left\{ q + \left[\max_{p \in \mathcal{Z}_p(w^- - q, t)} V((w^- - q)p, (w^- - q)(1-p), t_n^+) \right] \right\} \\ & \quad\; w^- = s + b \ . \end{aligned} \tag{6.3}$$

The optimal control $p_n(w)$ at time t_n is then determined from

$$p_n(w) = \begin{cases} \arg\max_{p' \in [0,1]} V(wp', w(1-p'), t_n^+), & w > 0 \; ; \; t_n \neq t_M \\ 0, & w \leq 0 \;\; \text{or } t_n = t_M \end{cases} . \tag{6.4}$$

The control for q is then determined from

$$q_n(w) = \arg\max_{q' \in \mathcal{Z}_q} \Big\{ q' + V((w-q')p_n(w-q')), (w-q')(1-p_n(w-q'))), t_n^+) \Big\} . \tag{6.5}$$

Remark 6.1 ($q_n(\cdot)$, $p_n(\cdot)$ control functions)**.** *From the right hand sides of equation (6.4) and equation (6.5), and noting equation (6.3), we can deduce the following results:*

(i) The optimal control for $q_n(\cdot)$ is a function only of the total portfolio wealth before withdrawals $w^- = s + b$, i.e. $q_n = q_n(w^-)$.

(ii) The optimal control for $p_n(\cdot)$ is a function only of the total portfolio wealth after withdrawals $w^+ = w^- - q_n(w^-)$, i.e. $p_n = p_n(w^+)$.

At $t = T$, we have

$$V(s, b, T^+) = \kappa \min((s + b - W^*), 0) . \tag{6.6}$$

For $t \in (t_{n-1}, t_n)$, there are no cash flows, discounting (all quantities are inflation adjusted), or controls applied, hence, for $h \to 0^+$,

$$V(s,b,t) = E\Big[V(S(t+h), B(t+h), t+h)\Big] \; ; \; t \in (t_{n-1}, t_n - h) . \tag{6.7}$$

Cognizant of processes (2.3) and (2.4), and using Ito's Lemma for jumps (Tankov and Cont, 2009), we follow the usual arguments to obtain

$$\begin{aligned} V_t &+ \frac{(\sigma^s)^2 s^2}{2} V_{ss} + (\mu^s - \lambda_\xi^s \kappa_\xi^s) s V_s + \lambda_\xi^s \int_{-\infty}^{+\infty} V(e^y s, b, t) f^s(y) \; dy + \frac{(\sigma^b)^2 b^2}{2} V_{bb} \\ &+ (\mu^b - \lambda_\xi^b \kappa_\xi^b) b V_b + \lambda_\xi^b \int_{-\infty}^{+\infty} V(s, e^y b, t) f^b(y) \; dy - (\lambda_\xi^s + \lambda_\xi^b) V + \rho_{sb} \sigma^s \sigma^b s b V_{sb} = 0 \quad , \\ & t \in (t_{n-1}, t_n) \quad . \end{aligned} \tag{6.8}$$

Remark 6.2 (Use of running sum of future withdrawals)**.** *Note that the objective function in equation (6.1) is written as (the expectation operator is understood)*

$$\textit{Objective Function} = \sum_{i=n}^{i=M} q_i + \kappa\Big(\min((W_T - W^*), 0)\Big) . \tag{6.9}$$

Instead of using the future running sum of withdrawals, an alternative would be average future withdrawals, i.e.

$$\textit{Alternative Objective Function} = \left(\frac{1}{M-n+1} \sum_{i=n}^{i=M} q_i\right) + \kappa'\Big(\min((W_T - W^*), 0)\Big) . \tag{6.10}$$

The optimal controls which maximize equation (6.10) will also maximize equation (6.9) if

$$\kappa' = \frac{\kappa}{M - n + 1} \ . \tag{6.11}$$

In other words, use of equation (6.9) (running sum) increases the weight on the risk term as $n \to M$ ($t \to T$), compared to equation (6.10) (average remaining cash flows). Intuitively, this makes sense. As the terminal time is approached, the investor puts greater emphasis on minimizing the risk of falling below the target.

6.1 Numerical algorithm: EW-LS

We use the dynamic programming formulation of the EW-LS problem $EWLS_{t_0}(\kappa))$ as outlined in equations (6.1-6.2). We discretize the state space (s,b), and solve PIDE (6.8) between rebalancing times using a Fourier method (Forsyth and Labahn, 2019).

We discretize the p controls and then solve the optimization problem (6.4) using exhaustive search over the discretized p values, linearly interpolating the right hand side discrete values of V in equation (6.4) as required. We also discretize the controls for q in the range $[q_{\min}, q_{\max}]$ in increments of one thousand dollars, and determine the optimal control for q by exhaustive search. We use a fixed discretization of the q controls since it is realistic to assume that retirees will change withdrawal amounts in fairly coarse increments.

We have carried out grid refinement studies (see Appendix C), which indicate that the PDE solution values (for EW and LS) have errors in the third digit (which is certainly accurate enough for practical purposes). We compute and store the controls in the synthetic market.

For reporting purposes, we then use the stored controls, and then carry out Monte Carlo simulations in the synthetic market. This allows us to generate a variety of statistics of interest. Similarly, we use stored controls, and also carry out bootstrap resampling simulations in the historical market.

6.1.1 Stabilization

If $W_t \gg W^*$, and $t \to T$, then $Pr[W_T < W^*] \simeq 0$. For large values of W_t, the withdrawal is capped at $q_{\max}$. In this case, the control only weakly effects the objective function. Although these states have low probability, it is desirable to enforce a particular choice of control. To this end, we changed the objective function in Problem 5.2 to

$$V\left(s,b,t_0^-\right) = \max_{\mathcal{P}_0 \in \mathcal{A}} \left\{ E_{\mathcal{P}_0}^{X_0^+,t_0^+} \left[\sum_{i=0}^{i=M} q_i \ + \ \kappa \Big(\min(W_T - W^*, 0) \Big) \overbrace{+\epsilon W_T}^{stabilization} \right. \right. \\ \left. \left. \Bigg| X(t_0^-) = (s,b) \right] \right\} . \tag{6.12}$$

We used the value $\epsilon = +10^{-6}$ in the following test cases. Using a positive value for ϵ has the effect of forcing the strategy to invest in stocks when W_t is very large, and $t \to T$, when the control problem is ill-posed. An alternative would be to use a negative value of ϵ, which would force the investor to allocate to an all bond portfolio for large values of wealth. However, it seems more reasonable to use a positive value for ϵ, which generates large possible values of wealth at age 80. Note that using this small value of $\epsilon = 10^{-6}$ gave the same results as $\epsilon = 0$ for the summary statistics, to four digits.

7 Investment scenario

Table 7.1 shows our base case investment scenario. We will use thousands as our units of wealth in the following. For example, a withdrawal of 40 per year corresponds to \$40,000 per year, with an initial wealth of 1000 (\$1,000,000). Thus, a withdrawal of 40 per year would correspond to the use of the four per cent rule (Bengen, 1994).

To make this example more concrete, this scenario would apply to a retiree who is 65 years old, with a pre-retirement salary of 100 (\$100,000) per year, with a total value of DC plan holdings at retirement of 1,000 (\$1,000,000). In Canada, a retiree would be eligible for government benefits (indexed) of about 20 per year. If the investor targets withdrawing 40 per year from the DC plan, then this would result in total real income of about 60 per year, which is about 60% of pre-retirement salary. The initial phase of the decumulation occurs for 15 years, taking the retiree to age 80.

For risk management purposes, we will assume that the retiree owns mortgage free real estate worth about 400, which will retain its value in real terms over 15 years. In a worst case scenario, we assume that the retiree can borrow 200 using a reverse mortgage, at age 80.

7.1 Data and calibration

In Appendix A, we give the details concerning the historical return data used, and the method for fitting the parameters for the parametric stock and bond processes (2.3) and (2.4). Briefly, the historical data is obtained from the Center for Research in Security Prices (CRSP) over the period 1926:1-2019:12. We check for the robustness of our results by testing the strategies determined based on the parametric model using bootstrap resampling of the historical data.

7.2 Choice of W^* for $\mathbf{LS}_{W^*}$

We fix W^* in equation (4.1) at the initial time. Let a_x be the present value of an annuity which pays one dollar per year (real), for the remaining lifetime of an x year old Canadian male. Given a portfolio value of W_T, then the lifetime annuity amount per year, which can be purchased with this wealth is W_T/a_x. Using a (pessimistic) value of real interest rates of zero, [4] then using the CPM2014 tables, we find that $1/a_{80} \simeq 0.10$. This suggests that $W_T = 400$ would generate 40 per year (all units: thousands) from a lifelong annuity purchased at age 80, with the implication that we should set $W^* = 400$.

Of course, setting $W^* = 400$ does not guarantee that $W_T \geq 400$, for any given value of κ in equation (5.1). In addition, fairly priced, real annuities are not available in the Canadian market. For example, as of October, 2020, a survey of online posted rates for a lifetime annuity for an 80 year old male (no guarantee, nominal dollars) resulted in $1/a_{80}$ in the range $0.087 - 0.097$.

In view of this, we should regard W^* as a parameter, and *post-hoc*, check other criteria to determine if the risk is acceptable, for a given value of κ. Assume that the CVAR(5%) of the portfolio at $t = T$ is > 300, and that the investor borrows 200 using his reverse mortgage (all units: thousands). In other words, the mean worst 5% of outcomes gives a total of 500 (real) at age 80. Using the best posted annuity rate gives $1/a_{80} = .097$, which results in a (nominal) payout of about 48.5 per year, which we assume to roughly equate to the target payout of 40 per year real. Clearly, there is significant uncertainty regarding the actual annuity payouts 15 years in the future, but it

[4] From Table A.1, we can see that the average real return of 10 year treasuries is about 0.0239. Using this interest rate, a fairly priced real annuity (using the CPM2014 tables) would have $1/a_{80} \simeq 0.11$.

seems that as long as CVAR(5%) + 200 $\geq$ 500, this suggests that the retiree can be reasonably sure[5] of purchasing a lifetime annuity which generates 40 real (even for the worst 5% of outcomes).

In the following numerical examples, we set $W^* = 400$ (units: thousands) in the definition of LS in equation (4.1). This fixed value of W^*, represents a desired minimum W_T, with a linear penalty for undershooting this amount. Varying the scalarization parameter κ in (5.1) traces out the EW-LS frontier. We will use the additional criteria that CVAR(5%) $\geq$ 300 (units: thousands), to select an appropriate point on the EW-LS frontier. This effectively specifies a suitable value of κ.

Investment horizon T (years)	15
Equity market index	CRSP Cap-weighted index (real)
Bond index	10-year Treasury (US) (real)
Initial portfolio value W_0	1000
Cash withdrawal times	$t = 0,1,\ldots,15$
Withdrawal range	$[q_{\min}, q_{\max}]$
Equity fraction range	[0,1]
Borrowing spread μ_c^b	0.0
Rebalancing interval (years)	1
Market parameters	See Table A.1

TABLE 7.1: *Input data for examples. Monetary units: thousands of dollars.*

7.3 Synthetic market

We fit the parameters for the parametric stock and bond processes (2.3 - 2.4) as described in Appendix A. We then compute and store the optimal controls based on the parametric market model. Finally, we compute various statistical quantities by using the stored control, and then carrying out Monte Carlo simulations, based on processes (2.3 - 2.4).

7.4 Historical market

We compute and store the optimal controls based on the parametric model (2.3-2.4) as for the synthetic market case. However, we compute statistical quantities with the stored controls, but using bootstrapped historical return data directly. We remind the reader that all returns are inflation adjusted. In Appendix B, we give details concerning the stationary block bootstrap resampling technique.

8 Synthetic and historical markets: constant withdrawals $q = 40$, constant proportion strategy

We consider the scenario in Table 7.1. As a benchmark, we consider withdrawing at a constant rate of 40 per year (units: thousands of dollars). This would correspond to the 4% rule suggested in (Bengen, 1994). We also assume that the portfolio is rebalanced to a constant weight in stocks each year. Recall that both bond and stocks follow jump diffusion processes. Hence, in the synthetic

[5] More precisely, the investor has enough wealth to purchase the desired annuity, as measured by the mean of the worst 5% of outcomes.

market, it is possible (although unlikely) that both stock and bond holdings can jump to zero, leaving the investor insolvent, without having the funds required for withdrawals, even including real estate. The same effect can also occur in the bootstrap resampling tests (i.e. repeated sampling from months with large stock drawdowns and high inflation). Hence, we focus on CVAR (5%) at the end of the first stage decumulation, as a reasonable risk measure, and not purely the worst case.

Table 8.1 shows the results for various equity weights in the synthetic market, while Table 8.2 shows results for the bootstrapped historical market.

The results are roughly comparable for both synthetic and historical markets. In both cases, the largest (best) value of $\text{LS}_{W^*} = E[\min((W_T - W^*),0)]$ and CVAR(5%) occurs at $p = 0.30$. Note that LS_{W^*} does not precisely track CVAR(5%), since the relationship (4.4) holds only if the W^* is the 5% VAR (see equation (4.3)). However, it appears that LS_{W^*} and CVAR(5%) do give similar risk rankings.

For both synthetic and historical markets, the best CVAR(5%) is about 300 (units thousands of dollars), at constant equity weight of $p = 0.30$. This meets our criteria of $\text{CVAR}(5\%) + 200 \geq 500$, which we estimate to be sufficient to purchase a real lifetime annuity of 40 per year, for an 80-year old male. Recall that we are not suggesting that the investor actually buys an annuity at $T = 15$ years, but that this constant equity weight strategy does meet our tail risk criteria.

Equity Weight	$Median[W_T]$	$E[\min(W_T - W^*,0)]$	CVAR (5%)
0.0	609.03	-20.051	181.12
0.20	818.26	-5.4987	293.65
0.30	922.24	-5.0422	299.54
0.40	1025.0	-6.1912	277.45
0.60	1223.4	-12.073	183.20

TABLE 8.1: *Synthetic market results assuming the scenario given in Table 7.1, with $q_{\max} = q_{\min} = 40$, and p_ℓ = constant in equation (5.3). Stock index: real capitalization weighted CRSP stocks; bond index: real 10-year US treasuries. Parameters from Table A.1. $W^* = 400$. Units: thousands of dollars. Statistics based on 2.56×10^6 Monte Carlo simulation runs.*

Equity Weight	$Median[W_T]$	$E[\min(W_T - W^*,0)]$	CVAR (5%)
0.0	612.56	-18.332	199.06
0.20	809.13	-5.0769	301.19
0.30	908.27	-4.4439	311.12
0.40	1007.2	-5.1714	296.62
0.60	1203.2	-9.4341	221.91

TABLE 8.2: *Historical market results (bootstrap resampling) assuming the scenario given in Table 7.1, except that $q_{\max} = q_{\min} = 40$, and p_ℓ = constant in equation (5.3). Stock index: real capitalization weighted CRSP stocks; bond index: real 10-year US treasuries. Historical data in range 1926:1-2019:12. Parameters from Table A.1. $W^* = 400$. Units: thousands of dollars. Statistics based on 10^5 bootstrap resampling simulations. Expected blocksize 0.25 years.*

9 Synthetic and historical markets: constant withdrawals $q = 35$, constant proportion strategy

Alternatively, we can reduce the constant withdrawal rate to 35 per year, for 15 years (up to age 80). The synthetic market results are shown in Table 9.1 and the bootstrapped historical results are given in Table 9.2 Again, the results are roughly comparable for both the synthetic and historical markets, with the largest value of $\text{LS}_{W^*} = E[\min((W_T - W^*),0)]$ occurring at $p = 0.3$. This constant equity value also generates the (best) largest value of CVAR. In this case, the largest value of CVAR is 382 for $p = 0.3$ (synthetic market) compared to CVAR(5%)= 394 for $p = 0.3$ in the historical market. These strategies comfortably meet our tail risk criteria of CVAR(5%) + 200 $\geq$ 500, but of course at the expense of smaller minimum withdrawals.

Note that using a constant withdrawal of $q = 35$ coupled with $p = 0.3$, while giving an acceptable result in terms of risk, has an undesirable spending pattern. The retiree has taken small withdrawals for the first 15 years. At the end of 15 years, the median remaining wealth is greater than 1000, meaning that the retiree can now increase spending in the years after 80, either by continuing to manage the portfolio or purchasing an annuity. Consequently, the constant proportion, constant withdrawal strategy is producing a spending pattern exactly the opposite of our objective, i.e. this strategy produces small spending before age 80, and increases spending after age 80.

Equity Weight	$Median[W_T]$	$E[\min(W_T - W^*,0)]$	CVAR (5%)
0.0	703.39	-9.0266	255.84
.20	923.11	-1.9584	374.97
.30	1032.3	-1.9461	382.05
.40	1140.0	-2.7371	360.12
.60	1348.4	-6.7687	264.64

TABLE 9.1: *Synthetic market results assuming the scenario given in Table 7.1, with $q_{\max} = q_{\min} = 35$, and p_ℓ = constant in equation (5.3). Stock index: real capitalization weighted CRSP stocks; bond index: real 10-year US treasuries. Parameters from Table A.1. $W^* = 400$. Units: thousands of dollars. Statistics based on 2.56×10^6 Monte Carlo simulation runs.*

Equity Weight	$Median[W_T]$	$E[\min(W_T - W^*,0)]$	CVAR (5%)
0.0	706.99	-7.5741	274.00
0.2	913.53	-1.7123	382.66
0.3	1017.7	-1.6249	393.72
0.4	1121.4	-2.1367	379.58
0.6	1325.9	-4.9388	303.68

TABLE 9.2: *Historical market results (bootstrap resampling) assuming the scenario given in Table 7.1, except that $q_{\max} = q_{\min} = 35$, and p_ℓ = constant in equation (5.3). Stock index: real capitalization weighted CRSP stocks; bond index: real 10-year US treasuries. Historical data in range 1926:1-2019:12. Parameters from Table A.1. $W^* = 400$. Units: thousands of dollars. Statistics based on 10^5 bootstrap resampling simulations. Expected blocksize 0.25 years.*

10 Synthetic market: efficient frontiers

In Appendix D, we give the detailed efficient EW-LS frontiers, computed in the synthetic market. The results are shown graphically in Figure 10.1, for both the cases $(q_{\min}, q_{\max}) = (40{,}60)$ and $(q_{\min}, q_{\max}) = (35{,}60)$. In Table 10.1, we show the detailed results for two specific points on the EW-LS curves.

κ	$E[\min(W - W^*,0)]$	$E[\sum_i q_i]/(M+1)$	CVAR (5%)	$Median[W_T]$	$Prob[W_T < W^*]$
		$q_{\min} = 40, q_{\max} = 60$			
20	-4.825	52.99	304.5	600.7	.0610
		$q_{\min} = 35, q_{\max} = 60$			
50	-1.880	51.51	362.6	626.3	.0294

TABLE 10.1: *Synthetic market results for optimal strategies, assuming the scenario given in Table 7.1. Stock index: real capitalization weighted CRSP stocks; bond index: ten year treasuries. More details in Appendix D. Parameters from Table A.1. Units: thousands of dollars. Statistics based on* 2.56×10^6 *Monte Carlo simulation runs. Control is computed using the Algorithm in Section 6.1, stored, and then used in the Monte Carlo simulations.* $(M+1)$ *is the number of withdrawals.* M *is the number of rebalancing dates.* $W^* = 400$. $\epsilon = 10^{-6}$.

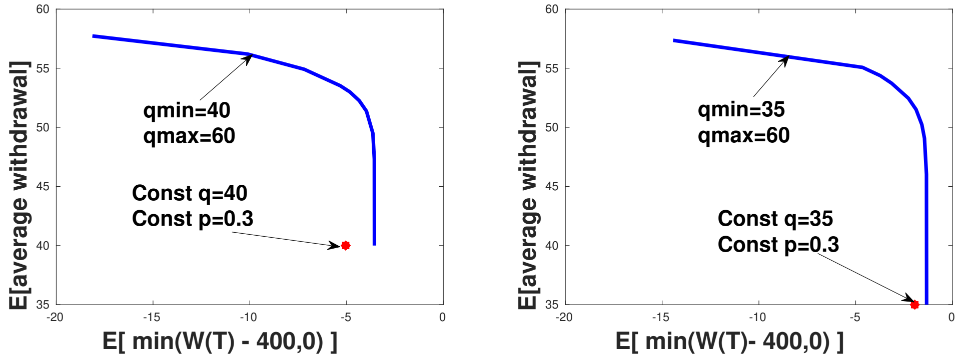
In Figure 10.1(a), we also show the point $(EW,LS) = (40, -5.04)$, the best result (in terms of LS risk) from Table 8.1 for the constant withdrawal, constant weight strategy with 40 per year.

This compares with the point $\kappa = 20$ in Table 10.1, for $(q_{\min}, q_{\max}) = (40{,}60)$, which is $(EW,LS) = (53, -4.83)$. In other words, at this point on the efficient frontier, the strategy generated by solving Problem 5.2 never withdraws less than 40 per year, has smaller tail risk (as measured by LS) and has an expected average withdrawal of 53. In terms of CVAR, the constant withdrawal strategy ($p = 0.3$) has CVAR(5%)= 300, compared with 305 for the optimal strategy. Note that $Pr[W_T \geq 400] \simeq 0.94$.

In Figure 10.1(b), we also show the point $(EW,LS) = (35, -1.95)$, which is the best result from Table 9.1 for $p = 0.30$. From Table 10.1, we can see that for $\kappa = 50$, $(q_{\min}, q_{\max}) = (35{,}60)$, the optimal strategy from Problem 5.2 generates $(EW,LS) = (51.5, -1.88)$. In other words, the optimal strategy never withdraws less than the constant withdrawal rate of 35, has a smaller risk as measured by LS, and an expected average withdrawal rate of 51.5. In terms of CVAR(5%), we see that the point $(EW,LS) = (51.5, -1.88)$ has a CVAR of 363, compared with the best constant weight, constant withdrawal strategy which has a CVAR of 382. In this case, this point on the efficient frontier has a larger (better) tail risk as measured by LS_{W^*}, and a slightly worse CVAR(5%) risk, but we never withdraw less than 35 per year, but with a much larger expected average withdrawal. We remind the reader that we are directly targeting risk as measured by LS in solving Problem 5.2, and that CVAR is not directly targeted, although CVAR and LS are related as in equations (4.3-4.4). In addition, we have that that $Pr[W_T \geq 400] \simeq 0.97$.

10.1 Synthetic market: optimal controls, withdrawals, wealth and heat map, $(q_{\min}, q_{\max}) = (40{,}60)$

In this section, we show the synthetic market results for the scenario in Table 7.1, with $(q_{\min}, q_{\max}) = (40{,}60)$, for the case $\kappa = 20$, $(EW,LS) = (53,-4.83)$. Figure 10.2 shows the percentiles of the optimal controls for the fraction in the stocks, the total wealth, and the withdrawals as a function of time. Up to about 10 years, the fraction in stocks (5^{th} to 95^{th} percentiles) is in the range $0.15 - 0.35$. The



(a) $q_{\min} = 40, q_{\max} = 60$.

(b) $q_{\min} = 35, q_{\max} = 60$.

FIGURE 10.1: *EW-LS frontiers. Scenario in Table 7.1. Optimal control computed from problem (5.2). Parameters based on real CRSP index, real 10-year US treasuries (see Table A.1). Control computed and stored from Problem 5.2 (synthetic market). Frontier computed using* 2.56×10^6 *MC simulations. Units: thousands of dollars.* $\epsilon = 10^{-6}$. $W^* = 400$.

median withdrawals are at the minimum for the first two years, and then increase to the maximum by year three.

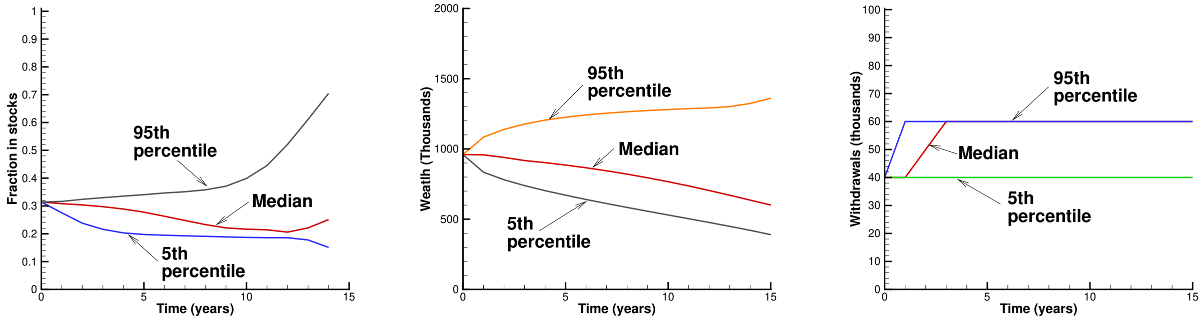
Figure 10.3 shows the heat maps for the optimal fraction in stocks and the optimal withdrawals. Note that the control for the fraction in stocks is shown as a function of wealth after withdrawals, and the control for the withdrawals is shown as a function of wealth before withdrawals (see Remark 6.1).

10.2 Historical market: optimal controls, withdrawals, wealth and heat map, $(q_{\min}, q_{\max}) = (35, 60)$

In this section, we compute and store the optimal controls in the synthetic market, for the scenario in Table 7.1, with $(q_{\min}, q_{\max}) = (35, 60)$, for the case $\kappa = 30$, with $(EW,LS) = (52, -1.87)$. These controls are then tested in the historical market, using 10^5 stationary block bootstrap resamples, with blocksize 0.25 years. The percentiles of the controls and wealth as a function of time are shown in Figure 10.4. Note that the additional flexibility of allowing smaller minimum withdrawals (35 compared to our target of 40), means that, the median withdrawal rate is at the maximum rate after the first year. The heat maps of the controls are shown in Figure 10.5.

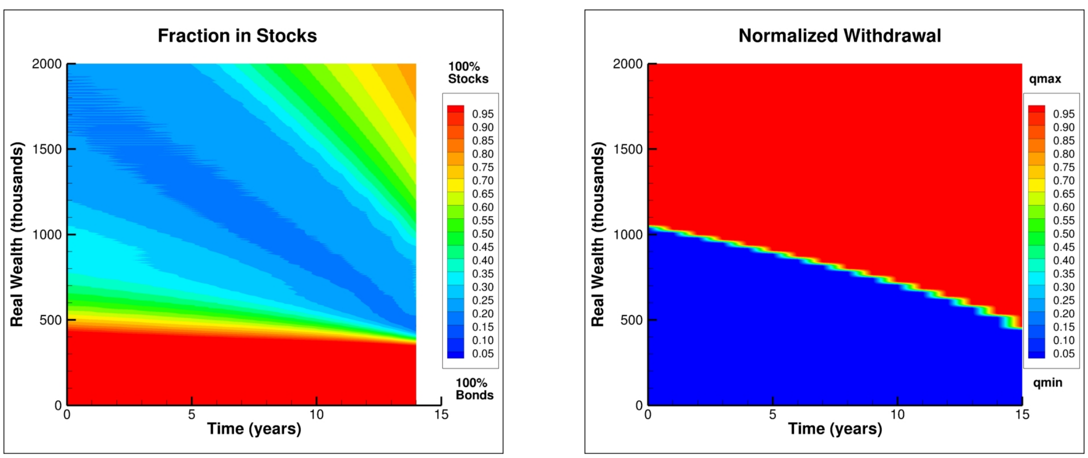
10.3 Bang-bang control for withdrawals

From Figure 10.3(b) and Figure 10.5(b) we can see that the optimal withdrawal, as a function of wealth before withdrawals, is either the maximum or minimum withdrawal, with a very small transition zone. This means that the optimal withdrawal is very close to a *bang-bang* type control. In Forsyth (2021), an analysis was carried out, assuming that the rebalancing interval tends to zero and that withdrawals occur at a continuous *rate* $\hat{q} \in [\hat{q}_{\min}, \hat{q}_{\max}]$. Note that the objective function in Forsyth (2021) is different from the objective function in this work, but the analysis of the continuously rebalanced problem is similar. Hence, it is possible to prove, in the continuous rebalancing limit, that the withdrawal controls are bang-bang, i.e. the optimal strategy is either to withdraw at the maximum rate $\hat{q}_{\max}$ or the minimum rate $\hat{q}_{\min}$. Of course, in our case, we have discrete rebalancing, and so the withdrawal control is not strictly bang-bang, but we can see from the heat maps that the control is very close to bang-bang.



(a) *Percentiles fraction in stocks* (b) *Percentiles wealth* (c) *Percentiles withdrawals*

FIGURE 10.2: *Scenario in Table 7.1. Optimal control computed from problem Problem 5.2. Scenario in Table 7.1. Parameters based on the real CRSP index, and real 10-year treasuries (see Table A.1). Control computed and stored from the Problem 5.2 in the synthetic market. Synthetic market,* 2.56×10^6 *MC simulations.* $q_{min} = 40, q_{\max} = 60$, $\kappa = 20$. $W^* = 400$. $\epsilon = 10^{-6}$. *Units: thousands of dollars.*



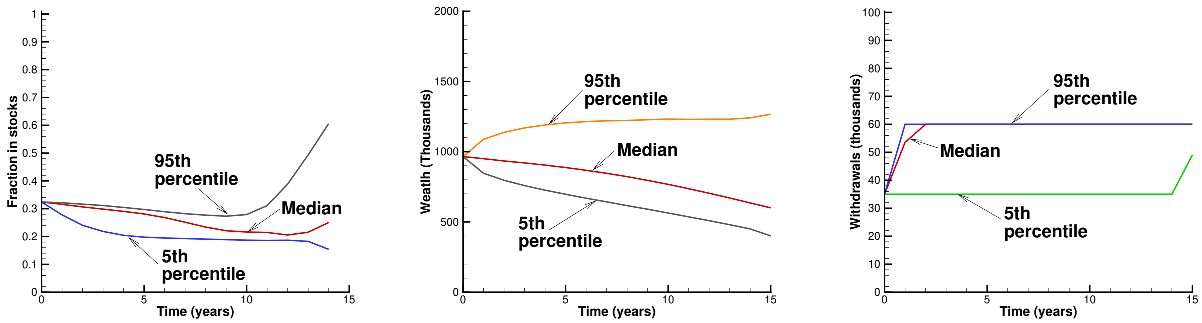
(a) *Fraction in stocks* (b) *Withdrawals*

FIGURE 10.3: *Heat map of controls: fraction in stocks and withdrawals, computed from Problem 5.2, cap-weighted real CRSP, real 10 year treasuries. Scenario given in Table 7.1. Control computed and stored from the Problem 5.2 in the synthetic market.* $q_{min} = 40, q_{\max} = 60$, $\kappa = 20$. $W^* = 400$. $\epsilon = 10^{-6}$. *Normalized withdrawal* $(q - q_{\min})/(q_{\max} - q_{\min})$. *Units: thousands of dollars.*

11 Robustness check: historical market

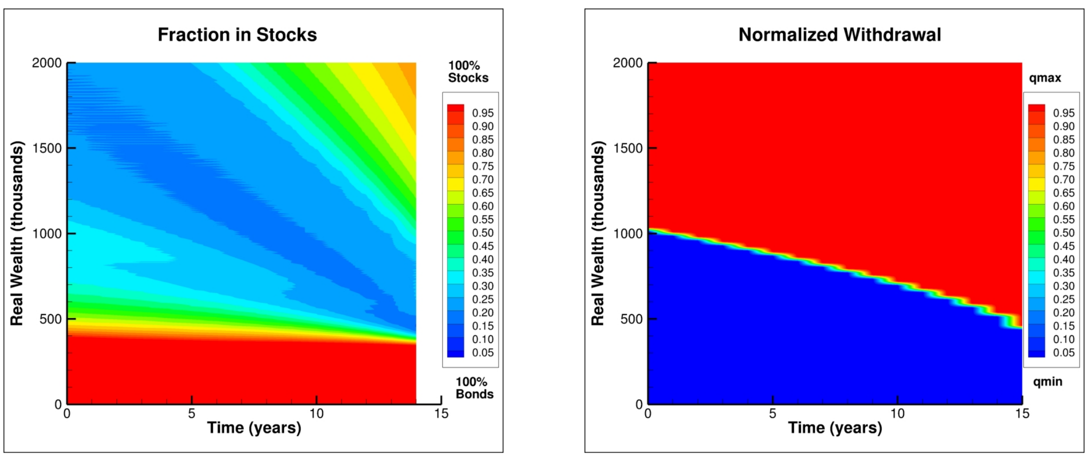
As a check on robustness of our results to parametric model misspecification, we carry out the following tests. The efficient EW-LS frontiers are computed in the synthetic market. The controls computed in the synthetic market are stored. These controls are used to construct the efficient EW-LS frontiers in the historical market, as well as in the synthetic market. The comparisons of the these frontiers are shown in Figure 11.1, for both the $(q_{\min}, q_{\max}) = (40{,}60)$ and $(q_{\min}, q_{\max}) = (35{,}60)$. The efficient frontiers for the synthetic and historical markets are very close, indicating that the controls computed in the synthetic market are robust to parametric model misspecification.

As mentioned previously, it is necessary to estimate an expected blocksize for use in the stationary block bootstrap resampling procedure. In Figure 11.2 we show the EW-LS frontiers, based on



(a) *Percentiles fraction in stocks* (b) *Percentiles wealth* (c) *Percentiles withdrawals*

FIGURE 10.4: *Scenario in Table 7.1. Optimal control computed from Problem 5.2. Scenario in Table 7.1. Parameters based on the real CRSP index, and real 10-year treasuries (see Table A.1). Control computed and stored from the Problem 5.2 in the synthetic market. Simulations carried out in the historical market, using 10^5 stationary block bootstrap resamples. Blocksize = 0.25 years. $q_{min} = 35, q_{\max} = 60$, $\kappa = 30$. $W^* = 400$. $\epsilon = 10^{-6}$. Units: thousands of dollars.*



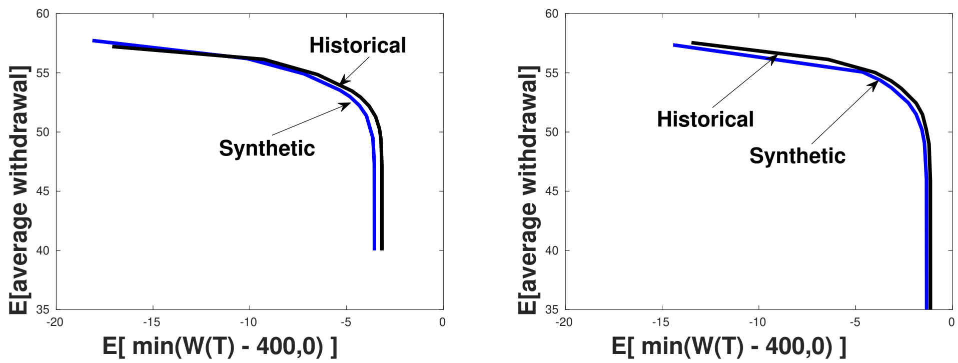
(a) *Fraction in stocks* (b) *Withdrawals*

FIGURE 10.5: *Heat map of controls: fraction in stocks and withdrawals, computed from Problem 5.2, cap-weighted real CRSP, real 10 year treasuries. Scenario given in Table 7.1. Control computed and stored from the Problem 5.2 in the synthetic market. $q_{min} = 35, q_{\max} = 60$, $\kappa = 30$. $W^* = 400$. $\epsilon = 10^{-6}$. Normalized withdrawal $(q - q_{\min})/(q_{\max} - q_{\min})$. Units: thousands of dollars.*

controls determined in the synthetic market, and tested in the historical market, computed using different blocksizes, for the case $(q_{\min}, q_{\max}) = (40{,}60)$. The detailed frontiers are given in Appendix E. The efficient frontiers are fairly robust to different choices of expected blocksize, varying from 0.25 to 1.0 years.

12 Discussion

We can see that allowing adaptive controls, both in the equity fraction and in the withdrawal amounts, improves the results considerably, compared to a constant weight, constant withdrawal strategy. For example, if we consider fixing the minimum withdrawal to be the same as the constant



(a) $q_{\min} = 40, q_{\max} = 60$. (b) $q_{\min} = 35, q_{\max} = 60$.

FIGURE 11.1: *EW-LS frontiers, comparison of synthetic frontiers, and frontier generated from (i) controls computed in the synthetic market (ii) control tested in the historical (bootstrapped) market. Scenario in Table 7.1. Parameters based on real CRSP index, real 10-year US treasuries (see Table A.1). Control computed and stored, historical frontier computed using* 10^5 *bootstrap resampled simulations, blocksize* 0.25 *years. Historical data in range 1926:1-2019:12. Units: thousands of dollars.* $W^* = 400$.

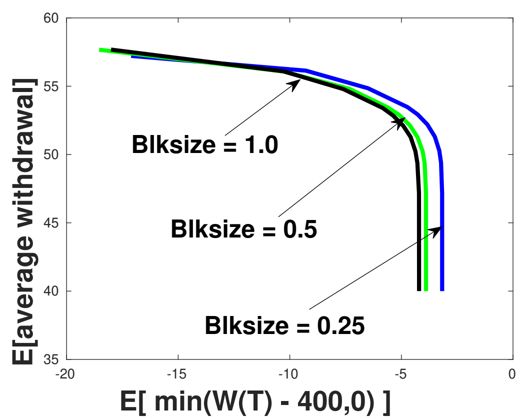


FIGURE 11.2: *EW-LS frontiers, comparison of frontier generated from (i) controls computed in the synthetic market (ii) control tested in the historical (bootstrapped) market. Scenario in Table 7.1.* $(q_{\min}, q_{\max}) = (40{,}60)$. *Parameters based on real CRSP index, real 10-year US treasuries (see Table A.1). Control computed and stored, historical frontier computed using* 10^5 *bootstrap resampled simulations, expected blocksizes as shown, in years.* $(q_{\min}, q_{\max}) = (40{,}60)$. *Historical data in range 1926:1-2019:12. Units: thousands of dollars.* $W^* = 400$.

withdrawal amount (assuming a constant weight equity strategy), and examine the point on the efficient frontier with similar risk, as measured by LS_{W^*}, we observe the following.

(i) The expected average withdrawal is considerably larger than the constant withdrawal amount.

(ii) The flexible withdrawal amount is never less than the fixed constant withdrawal amount.

(iii) By construction, the LS_{W^*} tail risk is the same or better than the constant weight, constant withdrawal policy. The CVAR(5%) risk is comparable.

In this sense, for practical purposes, the optimal policy, with stock fraction controls and flexible withdrawals, dominates the constant stock fraction, constant withdrawal strategy.

Our base scenario assumed a target withdrawal strategy of 40 per year (units thousands). If we allow more flexibility in the withdrawals, i.e. minimum of 35 per year, maximum 60 per year, then there are points on the efficient frontier with LS risk less than the constant weight, constant withdrawal strategy (35 per year), but with expected average withdrawals of more than 50 per year. This indicates that withdrawal flexibility can be used to both reduce risk and increase expected average withdrawals.

The optimal strategies directly targeted tail risk as measured by LS_{W^*} ($W^* = 400$). However, if we compare the various strategies after the fact, using CVAR(5%), our ranking of strategies is essentially the same. This suggests that these strategies are fairly robust to the particular measure of tail risk used.

Compared with previous results (Forsyth, 2021), (optimal strategy for longer investment horizons, objective function based on CVAR risk measure), we have increased withdrawals in the early years of retirement. These strategies have median withdrawals at the maximum withdrawal rates, within 2-3 years of retirement.

13 Conclusions

If we rule out the use of immediate lifelong annuities, then a major problem with decumulation strategies is uncertain longevity. One way around this is to specify a long decumulation period, i.e. 30 years. This is a conservative approach, but the optimal withdrawal controls (Forsyth, 2021) are such that the retiree withdraws as the minimum specified rate for $10 - 15$ years after retirement. This obviously reduces the risk of ruin before year 30 (i.e. before age 95 for a 65-year old retiree), but this is perhaps an undesirable spending pattern. In addition, of course, the probability of attaining age 95, is quite low. Therefore, it is probable that the retiree will pass away, leaving considerable wealth unspent.

At first sight, it might seem reasonable to use mortality weighted cash flows in the objective function. However, since the retiree does not actually purchase an annuity (by assumption), mortality credits are not actually earned. Hence, this does not produce the required minimum cash flows.[6]

As an alternative, in this work, we suggest initially examining a shorter decumulation horizon of 15 years. At the end of 15 years, we specify a tail risk target for the portfolio (including borrowing secured by real estate) that would be sufficient to purchase a lifelong annuity, at age 80, which would provide for minimum desired cash flows. Note that we are not suggesting that the retiree actually purchase an annuity at age 80. The annuity value is simply an appropriate tail risk target. Of course, the retiree's median wealth at age 80 will be considerably greater than this minimum target. In these cases, the retiree then has the flexibility to (i) continue on with the self-managed DC account or (ii) combine the DC account with an annuity. Note that the probability of a 65 year old Canadian male attaining the age of 80 is about 0.76, so that a 15 year decumulation period occurs with high probability.

The shorter investment horizon does indeed improve the spending pattern. The median withdrawal rate is at the maximum, within 2-3 years of retirement.

This suggests that breaking up the decumulation horizon into early and late portions is a desirable strategy. Focusing on maximizing total withdrawals during the early stage, while ensuring that the tail risk at 15 years is small, is both a low risk and high satisfaction policy. The tail risk

[6]A retiree at any point in time is either alive or dead. If alive, the retiree needs the full minimum cash flow, not the mortality weighted cash flow.

target is based on the fact that lifelong annuity payouts increase considerably by age 80, so this tail risk target represents a cost effective fall back strategy.

14 Acknowledgements

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Appendix

A Data

We use data from the Center for Research in Security Prices (CRSP) on a monthly basis over the 1926:1-2019:12 period.[7] Our tests use the CRSP 10 year US treasury index for the bond asset and the CRSP capitalization-weighted total return index for the stock asset. This latter index includes all distributions for all domestic stocks trading on major U.S. exchanges.[8] All of these various indexes are in nominal terms, so we adjust them for inflation by using the U.S. CPI index, also supplied by CRSP. We use real indexes since retirees should be focused on real (not nominal) cash flows.

We use the threshold technique (Mancini, 2009; Cont and Mancini, 2011; Dang and Forsyth, 2016) to estimate the parameters for the parametric stochastic process models, (2.3) and (2.4). Note that the data is inflation adjusted, so that all parameters reflect real returns. Table A.1 shows the results of calibrating the models to the historical data. The correlation ρ_{sb} is computed by removing any returns which occur at times corresponding to jumps in either series, and then using the sample covariance. Further discussion of the validity of assuming that the stock and bond jumps are independent is given in Forsyth (2020b).

An obvious generalization of processes (2.3) and (2.4) would be to include stochastic volatility effects. However, previous studies have shown that stochastic volatility appears to have little consequences for long term investors (Ma and Forsyth, 2016).

CRSP	μ^s	σ^s	λ^s	p^s_{up}	η^s_1	η^s_2	ρ_{sb}
	0.0877	0.1459	0.3191	0.2333	4.3608	5.504	0.04554
10-year Treasury	μ^b	σ^b	λ^b	p^b_{up}	η^b_1	η^b_2	ρ_{sb}
	0.0239	0.0538	0.3830	0.6111	16.19	17.27	0.04554

TABLE A.1: *Estimated annualized parameters for double exponential jump diffusion model. Value-weighted CRSP index, 10-year US treasury index deflated by the CPI. Sample period 1926:1 to 2019:12.*

[7] More specifically, results presented here were calculated based on data from Historical Indexes, ©2020 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

[8] The 10-year Treasury index was constructed from monthly returns from CRSP back to 1941. The data for 1926-1941 were interpolated from annual returns in Homer and Sylla (2005).

Data series	Optimal expected block size $\hat{b}$ (months)
Real 10-year Treasury index	4.2
Real CRSP capitalization-weighted index	3.1

TABLE B.1: *Optimal expected blocksize $\hat{b} = 1/v$ when the blocksize follows a geometric distribution $Pr(b = k) = (1 - v)^{k-1}v$. The algorithm in Patton et al. (2009) is used to determine $\hat{b}$. Historical data range 1926:1-2019:12.*

B Historical market: stationary block bootstrap resampling

We use the stationary block bootstrap method (Politis and Romano, 1994; Politis and White, 2004; Patton et al., 2009; Dichtl et al., 2016). A crucial parameter is the expected blocksize. Sampling the data in blocks accounts for serial correlation in the data series. We use the algorithm in Patton et al. (2009) to determine the optimal blocksize for the bond and stock returns separately, see Table B.1. We use a paired sampling approach to simultaneously draw returns from both time series. In this case, a reasonable estimate for the blocksize for the paired resampling algorithm would be about 0.25 years. Detailed pseudo-code for block bootstrap resampling is given in Forsyth and Vetzal (2019).

C Convergence test: synthetic market

We carry out an initial test of convergence of our numerical method for the EW-LS problem. We localize the problem to a grid with $(s,b) \in [s_{\min}, s_{\max}] \times [-b_{\max}, +b_{\max}]$, using artificial boundary conditions as discussed in (Forsyth and Labahn, 2019). We set $(s_{\min}, s_{\max}) = (100e^{-8}, 100e^{+8})$, and $b_{\max} = s_{\max}$. Increasing $s_{\max}$ by ten and decreasing $s_{\min}$ by ten resulted in no change to the solution to six figures. Table C.1 shows the results for solution of the PDE on a sequence of grids. For each refinement level, we store the optimal control, and use this control in Monte Carlo simulations. The PDE solution appears to converge at roughly a second order rate. In the following, we will report results based on (i) determining the control from the PDE solution (using the 2048×2048 grid in Table C.1) and (ii) using this control in Monte Carlo simulations. This allows us to generate various statistical quantities of interest.

D Detailed efficient frontiers: synthetic market

Tables D.1 and D.2 give the detailed results from the synthetic market used to construct Figure 10.1. Tables D.3 and D.4 give the details for the results computed in the historical market, used to construct Figure 11.1.

E Effect of blocksize

Tables E.1 and E.2 show the historical market results for the case $(q_{\min}, q_{\max}) = (40,60)$, using expected blocksizes of 0.5 and 1.0 years respectively. These tables can be compared to Table D.3 which uses an expected blocksize of 0.25 years.

	Algorithm in Section 6.1			Monte Carlo	
Grid	$E[(W_T - W^*)]$	$E[\sum_i q_i]/(M+1)$	Value Function	$E[(W_T - W^*)]$	$E[\sum_i q_i]/(M+1)$
512×512	-3.88020	49.4266	14.777	-3.61668	49.4139
1024×1024	-3.68539	49.5038	54.982	-3.61868	49.5044
2048×2048	-3.62496	49.5079	67.134	-3.61705	49.5033
4096×4096	-3.61089	49.5090	69.965	-3.61691	49.5067

TABLE C.1: *Convergence test, real stock index: deflated real capitalization weighted CRSP, real bond index: deflated ten year Treasuries. Scenario in Table 7.1. Parameters in Table A.1. The Monte Carlo method used* 10^7 *simulations.* $\kappa = 200$. *Grid refers to the grid used in the Algorithm in Section 6.1:* $n_x \times n_b$, *where* n_x *is the number of nodes in the* $\log s$ *direction, and* n_b *is the number of nodes in the* $\log b$ *direction. Units: thousands of dollars (real).* $(M+1)$ *is the total number of withdrawals.* M *is the number of rebalancing dates.* $q_{\min} = 40.0$. $q_{\max} = 60$. $W^* = 400.0$ *(Problem 5.2), Algorithm in Section 6.1.*

κ	$E[\min(W - W^*,0)]$	$E[\sum_i q_i]/(M+1)$	CVAR (5%)	$Median[W_T]$	$Prob[W_T < W^*]$
∞	-3.561	40.00	330.6	839.7	.0395
2000	-3.566	47.31	328.8	708.9	.0399
200	-3.617	49.50	327.3	661.2	.0420
50	-3.973	51.37	320.5	626.5	.0481
30	-4.343	52.25	313.2	612.3	.0539
20	-4.825	52.99	304.5	600.7	.0610
15	-5.328	53.52	296.0	529.3	.0676
10	-7.187	54.91	266.1	567.8	.0871
5	-10.12	56.20	224.5	554.6	.1150
2	-18.13	57.73	129.9	549.1	.1796

TABLE D.1: *Synthetic market results for optimal strategies, assuming the scenario given in Table 7.1. Stock index: real capitalization weighted CRSP stocks; bond index: ten year treasuries. Parameters from Table A.1. Units: thousands of dollars. Statistics based on* 2.56×10^6 *Monte Carlo simulation runs. Control is computed using the Algorithm in Section 6.1, stored, and then used in the Monte Carlo simulations.* $q_{\min} = 40.0$, $q_{\max} = 60$. $(M+1)$ *is the number of withdrawals.* M *is the number of rebalancing dates.* $W^* = 400$. $\epsilon = 10^{-6}$.

κ	$E[\min(W - W^*, 0)]$	$E[\sum_i q_i]/(M+1)$	CVAR (5%)	$Median[W_T]$	$Prob[W_T < W^*]$
∞	-1.334	35.00	399.1	932.1	.0179
2000	-1.342	46.02	383.7	733.1	.0182
200	-1.445	49.07	375.2	669.8	.0208
100	-1.581	50.23	369.8	648.8	.0236
50	-1.880	51.51	362.6	626.3	.0294
30	-2.277	52.46	354.6	610.2	.0364
20	-3.143	53.73	337.2	597.1	.0485
15	-3.727	54.37	325.8	573.6	.0566
10	-4.644	55.07	309.6	564.4	.0688
2	-14.43	57.36	171.4	551.6	.1643

TABLE D.2: *Synthetic market results for optimal strategies, assuming the scenario given in Table 7.1. Stock index: real capitalization weighted CRSP stocks; bond index: ten year treasuries. Parameters from Table A.1. Units: thousands of dollars. Statistics based on* 2.56×10^6 *Monte Carlo simulation runs. Control is computed using the Algorithm in Section 6.1, stored, and then used in the Monte Carlo simulations.* $q_{\min} = 35.0$, $q_{\max} = 60$. $(M+1)$ *is the number of withdrawals.* M *is the number of rebalancing dates.* $W^* = 400$. $\epsilon = 10^{-6}$.

κ	$E[\min(W - W^*, 0)]$	$E[\sum_i q_i]/(M+1)$	CVAR (5%)	$Median[W_T]$	$Prob[W_T < W^*]$
∞	-3.174	40.0	338.5	829.1	.0387
2000	-3.175	47.17	336.6	701.2	.0390
200	-3.223	49.40	335.6	653.3	.0403
100	-3.302	50.28	334.0	636.2	.0424
50	-3.500	51.30	330.00	617.5	.0469
30	-3.833	52.19	323.4	602.5	.0532
20	-4.271	52.93	315.4	590.5	.0602
15	-4.727	53.47	307.8	582.2	.0676
10	-6.497	54.85	279.7	556.8	.0881
5.0	-9.282	56.14	241.2	542.6	.1175
2.0	-17.11	57.22	150.5	535.9	.1874

TABLE D.3: *Control computed in the synthetic market, assuming the scenario given in Table 7.1. Stock index: real capitalization weighted CRSP stocks; bond index: ten year treasuries. Parameters from Table A.1. Units: thousands of dollars. Statistics based on* 10^5 *bootstrap resampling of the historical data. Historical data in range 1926:1-2019:12. Expected blocksize* $\hat{b} = .25$ *years.* $q_{\min} = 40.0$, $q_{\max} = 60$. $(M+1)$ *is the number of withdrawals.* M *is the number of rebalancing dates.* $W^* = 400$.

κ	$E[\min(W - W^*,0)]$	$E[\sum_i q_i]/(M+1)$	CVAR (5%)	$Median[W_T]$	$Prob[W_T < W^*]$
∞	-1.136	35.0	405.3	928.2	.0166
2000	-1.140	45.90	389.6	725.7	.0167
200	-1.210	49.00	380.8	662.0	.0188
100	-1.348	50.18	375.6	640.4	.0217
50	-1.547	51.48	369.3	617.3	.0272
30	-1.869	52.43	362.8	600.3	.0343
20	-2.638	53.68	347.3	574.3	.0469
15	-3.167	54.32	336.9	563.1	.0534
10	-4.008	55.03	322.2	553.2	.0690
5.0	-6.426	56.13	284.4	541.9	.0985
2.0	-13.48	57.55	190.1	538.0	.1714

TABLE D.4: *Control computed in the synthetic market, assuming the scenario given in Table 7.1. Stock index: real capitalization weighted CRSP stocks; bond index: ten year treasuries. Parameters from Table A.1. Units: thousands of dollars. Statistics based on* 10^5 *bootstrap resampling of the historical data. Historical data in range 1926:1-2019:12. Expected blocksize* $\hat{b} = .25$ *years.* $q_{\min} = 35.0$, $q_{\max} = 65$. $(M+1)$ *is the number of withdrawals.* M *is the number of rebalancing dates.* $W^* = 400$.

κ	$E[\min(W - W^*,0)]$	$E[\sum_i q_i]/(M+1)$	CVAR (5%)	$Median[W_T]$	$Prob[W_T < W^*]$
∞	-3.896	40.00	322.6	830.2	.0447
2000	-3.898	47.19	322.1	701.9	.0451
200	-3.958	49.37	320.9	654.7	.0465
100	-4.051	50.25	318.9	637.8	.0489
50	-4.274	51.26	314.6	619.6	.0536
30	-4.622	52.14	308.4	605.4	.0597
20	-5.086	52.88	300.6	593.2	.0671
15	-5.591	53.43	292.6	585.0	.0742
10	-7.331	54.81	264.5	559.6	.0942
5.0	-10.32	56.09	225.7	544.4	.1236
2.0	-18.54	57.67	133.3	536.9	.1905

TABLE E.1: *Control computed in the synthetic market, assuming the scenario given in Table 7.1. Stock index: real capitalization weighted CRSP stocks; bond index: ten year treasuries. Parameters from Table A.1. Units: thousands of dollars. Statistics based on* 10^5 *bootstrap resampling of the historical data. Historical data in range 1926:1-2019:12. Expected blocksize* $\hat{b} = .5$ *years.* $q_{\min} = 40.0$, $q_{\max} = 60$. $(M+1)$ *is the number of withdrawals.* M *is the number of rebalancing dates.* $W^* = 400$.

κ	$E[\min(W-W^*,0)]$	$E[\sum_i q_i]/(M+1)$	CVAR (5%)	$Median[W_T]$	$Prob[W_T < W^*]$
∞	-4.208	40.00	315.9	831.7	.0488
2000	-4.212	47.19	315.6	703.1	.0493
200	-4.283	49.37	314.4	656.1	.0513
100	-4.370	50.23	312.7	639.2	.0526
50	-4.586	51.28	309.0	621.2	.0591
30	-4.921	52.11	303.6	606.8	.0656
20	-5.371	52.85	296.7	594.1	.0728
15	-5.832	53.40	290.0	585.5	.0795
10	-7.600	54.79	264.1	559.4	.0993
5.0	-10.28	56.09	228.3	544.8	.1247
2.0	-18.01	57.68	138.3	534.6	.1896

TABLE E.2: *Control computed in the synthetic market, assuming the scenario given in Table 7.1. Stock index: real capitalization weighted CRSP stocks; bond index: ten year treasuries. Parameters from Table A.1. Units: thousands of dollars. Statistics based on* 10^5 *bootstrap resampling of the historical data. Historical data in range 1926:1-2019:12. Expected blocksize* $\hat{b} = 1.0$ *years.* $q_{\min} = 40.0$, $q_{\max} = 60$. $(M+1)$ *is the number of withdrawals.* M *is the number of rebalancing dates.* $W^* = 400$.

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