Optimal Trade Execution: A Mean–Quadratic-Variation Approach *

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Abstract

We propose the use of a mean–quadratic-variation criteria to determine an optimal trading strategy in the presence of price impact. We derive the Hamilton Jacobi Bellman (HJB) Partial Differential Equation (PDE) for the optimal strategy, assuming the underlying asset follows Geometric Brownian Motion (GBM) or Arithmetic Brownian Motion (ABM). The exact solution of the ABM formulation is in fact identical to the static (price-independent) approximate solution for the mean-variance objective function in Almgren and Chriss (2001). The optimal trading strategy in the GBM case is in general a function of the asset price. The static strategy determined in the ABM formulation turns out to be an excellent approximation for the GBM case, even when volatility is large.

Keywords: optimal trading, mean–quadratic-variation, HJB equation

JEL Classification: C63, G11

1 Introduction

A typical problem faced by an investment bank arises when buying or selling a large block of shares. If the trade is executed rapidly, then this can be expected to cause a significant price impact. For example, in the case of selling, this price impact will lower the average price received per share compared to the pretrade price. An obvious strategy is to break up the trade into a set of smaller blocks. This will lower the price impact, but now the trading takes place over a longer time horizon. Consequently, the seller is exposed to risk due to the stochastic movement of the stock price, relative to the pretrade price.

Algorithmic trading strategies attempt to determine a trading schedule which optimizes a given objective function. One of the early papers on this topic (Bertsimas and Lo, 1998) considered the best trading strategy which minimizes the cost of trading over a fixed time. More recently, this problem has been posed in terms of a mean-variance tradeoff in continuous time (Almgren and Chriss, 2001; Almgren, 2003; Almgren et al., 2004; Engle and Ferstenberg, 2007; Lorenz, 2008; Lorenz and Almgren, 2011). Another possibility is to maximize an exponential or power law utility function (He and Mamaysky, 2005; Vath et al., 2007; Schied and Schoneborn, 2008). However, the mean-variance tradeoff has a simple intuitive interpretation, and is probably preferred by practitioners.

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While single period mean-variance optimization is straightforward to understand, multi-period mean-variance optimization is more complex. In particular, there are several formulations of multi-period mean-variance optimization in the literature. The formulation that aguably aligns best with performance measures used in practice is the mean-variance criteria as seen at the start of the trade execution. This formulation is termed the pre-commitment mean-variance formulation in the literature. (Basak and Chabakauri, 2010)

One subtlety of the pre-commitment mean-variance formulation is that it gives rise to optimal strategies that are time-inconsistent. Consequently, the dynamic programming principle cannot be directly applied to solve for the optimal strategies. In view of this difficulty, various approximations to the pre-commitment mean-variance formulation have been proposed. Our work is a step toward understanding how such approximate strategies compare with the the truly optimal strategies.

In particular, we offer a new perspective on the Almgren and Chriss (2001) approximation. The approach in Almgren and Chriss (2001) restricts attention to static strategies (i.e. do not use any information regarding the stock price evolution after the start of trading). For static strategies, it turns out that variance, which is the original risk measure intended to be minimized, becomes equal to the expected value of quadratic variation. Nevertheless, we emphasize that this equality, by itself, does not imply that the static solution in Almgren and Chriss (2001) is optimal for the quadratic variation risk measure. The subtle point here is that Almgren and Chriss (2001) does not study how dynamic strategies perform in terms of the quadratic variation risk measure.

We thus suggest directly formulating the optimal trade execution problem in terms of a mean quadratic variation objective function. We discuss the conceptual simplicity of the mean quadratic variation formulation, and put forward the idea that quadratic variation is a sensible risk measure. We also discuss the properties of the optimal strategies under this risk measure. Our earlier paper (Forsyth, 2011) focuses on solving for the truly optimal solutions for the pre-commitment mean variance formulation. In addition, (Tse et al., 2011) compares the optimal strategies determined by the pre-commitment mean variance formulation and the mean quadratic variation formulation.

In summary:

- The mean quadratic variation formulation is conceptually simpler to understand than the pre-commitment mean variance formulation. In particular the mean quadratic variation objective leads to optimal strategies that are clearly time-consistent and can be easily obtained using the dynamic programming principle. In contrast, the pre-commitment mean variance formulation has optimal strategies that are time-inconsistent, and these strategies are considered unnatural by some authors (Basak and Chabakauri, 2010).

- As a risk measure, quadratic variation has the property that it takes into account the trajectory of the liquidation profile, as opposed to variance which measures only the end result with no concern of how liquidation proceeds during the whole trading horizon. Alternatively, minimizing quadratic variation can also be seen as minimizing the volatility of the portfolio value. Note that one purpose of converting shares to cash is to make the portfolio value process less volatile.

We also mention that Gatheral and Schied (2010) suggest time averaged value at risk as another risk measure that also leads to time-consistent strategies.

The main contributions of this article are

- We formulate the mean quadratic variation problem in which quadratic variation, rather than variance, is used as the risk measure. We argue that quadratic variation can be regarded as a reasonable risk measure in optimal trade execution. We derive the Hamilton Jacobi Bellman (HJB) partial differential equations (PDE) and provide numerical methods to solve for both the optimal strategies and the efficient frontier, with arbitrary constraints on the strategy.

- We formulate the optimal trade execution problem assuming that the asset price dynamic follows either Geometric Brownian Motion (GBM) or Arithmetic Brownian Motion (ABM). We believe that GBM is a superior model since it can be used for both long and short trading horizons, and avoids the fallacy of negative price scenarios that can appear in ABM. Nevertheless, we also study the ABM case since this allows us to compare our results with those in Almgren and Chriss (2001).

- We provide a proof that the classic static solution in Almgren and Chriss (2001) is optimal under the mean quadratic variation formulation, even when optimization is over the class of dynamic strategies (in the ABM case). The static solution in Almgren and Chriss (2001) is originally obtained as an approximate solution to the pre-commitment mean variance problem by restricting attention to static strategies. As such, the static strategy is strictly suboptimal in the pre-commitment mean variance formulation, since the optimal strategies are dynamic (Forsyth, 2011; Lorenz and Almgren, 2006; Lorenz, 2008). Therefore, our proof shows that the classic static solution in Almgren and Chriss (2001) is actually an exact optimal solution to the mean quadratic variation problem assuming ABM.

- We show that in the mean quadratic variation formulation, optimal strategies in the GBM case are qualitatively different from those in the ABM case. More specifically, optimal strategies are dynamic in the GBM case, whereas optimal strategies are static in the ABM case. This contrasts with the pre-commitment mean variance formulation in which optimal strategies are dynamic in both the GBM case and the ABM case.

- Our numerical results show that if we use the optimal static strategies from the ABM case as approximate solutions for the GBM case, this results in an efficient frontier that is very close to the true efficient frontier. While the accuracy of this ABM approximation is obvious when the ABM and the GBM dynamics are close, it is surprising that the accuracy of the ABM approximation is excellent even when volatility is very large. We explain this in detail and note that the accuracy of this approximation does not hold in the pre-commitment mean variance formulation.

2 Optimal Execution

Let

$$\begin{aligned} P &= B + AS = \text{ Portfolio} \\ S &= \text{ Price of the underlying risky asset,} \\ B &= \text{ Balance of risk free bank account,} \\ A &= \text{ Number of shares of underlying asset.} \end{aligned}$$

The optimal execution problem over $t \in [0, T]$ has the initial condition

$$S(0) = s_{init}, B(0) = 0, A(0) = \alpha_{init}. \tag{2.1}$$

If $\alpha_{init} > 0$, the trader is liquidating a long position (selling). If $\alpha_{init} < 0$, the trader is liquidating a short position (buying). In this article, for definiteness, we consider the selling case. At $t = T$,

$$S = S(T), B = B(T), A(T) = 0, \tag{2.2}$$

where $B(T)$ is the cash generated by selling shares and investing in the risk free bank account B, with a final liquidation at $t = T^-$ to ensure that $A(T) = 0$. The objective of optimal execution is to maximize $B(T)$, while at the same time minimizing a certain risk measure.

In this paper, we consider Markovian trading strategies $v(\cdot)$ that specify a trading rate v as a function of the current state, i.e. $v(\cdot) : (S(t), A(t), t) \mapsto v = v(S(t), A(t), t)$. Note that in using the shorthand notations $v(\cdot)$ for the mapping, and v for the value $v = v(S(t), A(t), t)$, the dependence of v on the current state is implicit.

By definition,

$$dA(t) = v \; dt. \tag{2.3}$$

We assume that due to temporary price impact, selling shares at the rate v at the market price $S(t)$ gives an execution price $S_{exec}(v,t) \leq S(t)$. It follows that

$$dB(t) \quad = \quad \big(rB(t) - vS_{exec}(v,t)\big)dt \tag{2.4}$$

where r is the risk free rate.

We suppose that the market price of the risky asset S follows a Geometric Brownian Motion (GBM) or arithmetic Brownian Motion (ABM), where the drift term is modified due to the permanent price impact of trading.

In the GBM model, we assume

$$\begin{aligned} dS(t) \quad &= \quad \big(\mu + g(v)\big)S(t)\ dt + \sigma S(t)\ d\mathbb{W}(t), \\ & \quad\quad \mu = \text{ drift rate}, \\ & \quad\quad g(v) \text{ is the permanent price impact function}, \\ & \quad\quad \sigma = \text{ volatility}. \\ & \quad\quad \mathbb{W}(t) \text{ is a Wiener process under the real world measure}. \end{aligned} \tag{2.5}$$

In the ABM model, we assume

$$dS(t) \quad = \quad \big(\mu + g(v)\big)S(0)\ dt + \sigma S(0)\ d\mathbb{W}(t)\ . \tag{2.6}$$

3 Price Impact

In this section we specify the permanent and temporary price impact functions used in this paper. We refer the reader to Almgren et al. (2004) for a discussion of the rationale behind these permanent and temporary price impact models.

In both the GBM case and the ABM case, we use the following form for the permanent price impact

$$\begin{aligned} g(v) \quad &= \quad \kappa_p v, \\ & \quad\quad \kappa_p \text{ is the permanent price impact factor }\ . \end{aligned} \tag{3.1}$$

We take κ_p to be a constant. This form of permanent price impact eliminates round-trip arbitrage opportunities, as discussed in Appendix B.

3.1 Geometric Brownian Motion

In the GBM case, we assume the temporary price impact scales linearly with the asset price, i.e.

$$S_{exec}(v,t) = f(v)S(t), \tag{3.2}$$

where

$$\begin{aligned} f(v) \quad &= \quad [1 + \kappa_s \operatorname{sgn}(v)] \exp[\kappa_t \operatorname{sgn}(v)|v|^{\beta}], \\ & \quad\quad \kappa_s \text{ is the bid-ask spread parameter },\\ & \quad\quad \kappa_t \text{ is the temporary price impact factor}, \\ & \quad\quad \beta \text{ is the price impact exponent }\ . \end{aligned} \tag{3.3}$$

Note that we assume $\kappa_s < 1$, so that $S_{exec}(v,t) \geq 0$, regardless of the magnitude of v.

3.2 Arithmetic Brownian Motion

In the GBM case, we assume the temporary price impact is asset-price-independent, i.e.

$$S_{exec}(v,t) = S(t) + S(0)h(v), \tag{3.4}$$

where

$$h(v) = \kappa_s \operatorname{sgn}(v) + \kappa_t v \ , \tag{3.5}$$

to be in accordance with Almgren and Chriss (2001). Note that $S_{exec}(v,t)$ may be negative for $v \to -\infty$, i.e. (3.5) is only valid for small trading rates.

Temporary impact (3.4), (3.5) is related to (3.2), (3.3) as follows. Assuming $\beta = 1$, $\kappa_t|v| \ll 1$ and $\kappa_t\kappa_s \ll 1$, temporary impact of the form (3.3) is approximately

$$f(v) \approx 1 + \kappa_s \operatorname{sgn}(v) + \kappa_t v \tag{3.6}$$

by Taylor's expansion. Therefore, the price impact (in dollar terms) is

$$S_{exec}(v,t) - S(t) = f(v)S(t) - S(t) = S(t)[\kappa_s \operatorname{sgn}(v) + \kappa_t v] \approx S(0)[\kappa_s \operatorname{sgn}(v) + \kappa_t v]$$

for $S(t) \approx S(0)$.

4 Definition of liquidation value

Given the state $(S(T^-), A(T^-))$ at the instant $t = T^-$ before the end of the trading horizon, we have one final liquidation (if necessary) so that the number of shares owned at $t = T$ is $A(T) = 0$. The liquidation value $B(T)$ after this final trade is defined to be[1]

$$\begin{aligned} B(T) &= B(T^-) + \lim_{v\to-\infty} A(T^-)S_{exec}(v,T^-) \\ &= \int_0^{T^-} e^{r(T-t')}\big(-vS_{exec}(v,t')\big)\, dt' + \lim_{v\to-\infty} A(T^-)S_{exec}(v,T^-). \end{aligned} \tag{4.1}$$

In the GBM case, applying (3.2) and (3.3) to (4.1) gives

$$B(T) = B(T^-). \tag{4.2}$$

In the ABM case, applying (3.4) and (3.5) to (4.1) gives

$$\begin{aligned} B(T) &= B(T^-) \quad &;\ A(T^-) = 0 \\ &= -\infty \quad &;\ \text{otherwise} \end{aligned} \tag{4.3}$$

Definition (4.1) in effect penalizes the strategy if $A(T) \neq 0$, so that the optimal algorithm forces the liquidation profile towards $A(T) = 0$. In the GBM case (4.2), the penalty is such that the shares $A(T^-)$ are simply discarded. In the ABM case (4.3), a large penalty is imposed.[2]

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[1] Note that we adopt the convention that $B(0) = 0$; see (2.1).

[2] In actual implementation, we would replace $\lim_{v\to-\infty}$ by a finite $v_{\min} \ll 0$ in the PDE initial condition. Also, in the case of liquidating a short position (buying), which is not considered in this paper, equation (4.1) would be defined as $B(T) = B(T^-) + \lim_{v\to\infty} A(T^-)S_{exec}(v,T^-)$, and we would replace $\lim_{v\to\infty}$ by a finite $v_{\max} \gg 0$ in implementation.

5 Mean Quadratic Variation formulation

Quadratic variation has been used as an approximation of variance in the algorithmic trading literature (Almgren and Chriss, 2001; Lorenz and Almgren, 2006). This approximation, however, can be poor when the trading impact is relatively large, as explained in Appendix C and illustrated in Tse et al. (2011). Instead of using quadratic variation to approximate variance, it is conceptually simpler to regard quadratic variation as an alternative risk measure.

5.1 Quadratic variation as a risk measure

Formally, the quadratic variation risk measure is defined as

$$E\left[\int_t^T \big(A(t')dS(t')\big)^2\right]. \tag{5.1}$$

Informally, the risk measure definition (5.1) can be interpreted as the quadratic variation of the portfolio value process as follows: by expanding the square of $dP(t') = dB(t') + d(A(t')S(t'))$ and ignoring higher-order terms, we have

$$\int_t^T \big(A(t')dS(t')\big)^2 = \int_t^T \big(dP(t')\big)^2, \tag{5.2}$$

when the trading velocity process $v(t)$ is bounded.

From the interpretation (5.2), minimizing quadratic variation clearly corresponds to minimizing volatility in the portfolio value process. The definition (5.1) shows that quadratic variation takes into account the trading trajectory $A(t')$ over the whole trading horizon. This is in contrast with using variance ($Var[B(T)]$) as a risk measure, which is independent of the trading trajectory $A(t')$ given the end result $B(T)$. We note that the idea of using quadratic variation as a risk measure was first suggested in Brugiere (1996).

5.2 Objective functional and value function

Now we specify the mean quadratic variation formulation as follows. For a fixed initial point $(s, \alpha, t) = (S(t), A(t), t)$ where $t < T$ with $B(t) = 0$, trading strategy $v(\cdot)$, and risk aversion parameter λ, we define the objective functional

$$J(s, \alpha, t, v(\cdot); \lambda) = E_{v(\cdot)}^{s,\alpha,t}\Big[B(T)\Big] - \lambda E_{v(\cdot)}^{s,\alpha,t}\left[\int_t^T \big(A(t')dS(t')\big)^2\right] \tag{5.3}$$

where

$$B(T) = \int_t^{T^-} e^{r(T-t')}\big(-vS_{exec}(v,t')\big)dt' + \lim_{v\to-\infty} A(T^-)S_{exec}(v,T^-) \tag{5.4}$$

and $E_{v(\cdot)}^{s,\alpha,t}[\cdot]$ is the conditional expectation at the initial point (s, α, t) using the control $v(\cdot)$.

The value function $\hat{V}$ is defined as

$$\hat{V}(s, \alpha, t; \lambda) \quad = \quad \sup_{v(\cdot)} J(s, \alpha, t, v(\cdot); \lambda). \tag{5.5}$$

For a given initial state (s, α, t), we will henceforth use the notation $v^*_{s,\alpha,t,\lambda}(\cdot)$ to denote the optimal policy that maximizes the corresponding functional, i.e. $J(s, \alpha, t, v(\cdot); \lambda)$.

5.3 Time Consistency of the optimal strategies

Let (s_1, α_1, t_1) be some state at time t_1 and $v^*_{s_1,\alpha_1,t_1,\lambda}(\cdot)$ be the corresponding optimal strategy. Let (s_2, α_2, t_2) be some other state at time $t_2 > t_1$ and $v^*_{s_2,\alpha_2,t_2,\lambda}(\cdot)$ be the corresponding optimal strategy.[3]

Since the optimal controls satisfy the Bellman's principle of optimality as shown in Appendix A, it follows that the optimal controls of (5.5) are *time consistent* in the sense that for the same state (s', α', t') at a later time $t' > t_2$,

$$v^*_{s_1,\alpha_1,t_1,\lambda}(s', \alpha', t') \quad = \quad v^*_{s_2,\alpha_2,t_2,\lambda}(s', \alpha', t') \ ; \ t' \geq t_2 \quad . \tag{5.6}$$

In view of equation (5.6), we can drop the subscript and just write $v^*(\cdot)$.

6 HJB Equation Formulation: GBM case

For $t < T$, let $V = V(s, \alpha, \tau = T - t; \lambda) = \hat{V}(s, \alpha, t; \lambda)$. For notational simplicity, we drop the parameter λ from V henceforth, i.e. we simply write $V = V(s, \alpha, \tau)$. Unless otherwise stated, we also restrict the admissible controls $v(\cdot)$ to be non-positive (i.e. only selling is permitted).

6.1 Optimal Control

The optimal control $v^*(\cdot)$ can be obtained by solving the following HJB PDE derived in Appendix A:

$$V_\tau \quad = \quad \mu s V_s + \frac{\sigma^2 s^2}{2} V_{ss} - \lambda \sigma^2 \alpha^2 s^2 + \sup_{v \leq 0} \left[e^{r\tau}(-v f(v))s + g(v) s V_s + v V_\alpha \right] . \tag{6.1}$$

Note that V has so far been defined for $\tau > 0$ only. Section 4 suggests that the initial condition for $V(s, \alpha, \tau = 0)$ should be

$$V(s, \alpha, \tau = 0) = \lim_{v \to -\infty} \alpha s f(v) \tag{6.2}$$

for the GBM model with temporary price impact (3.2).

6.2 Expected Value

In order to construct the efficient frontier, i.e. a plot of expected gain versus risk, we will need to compute the expected gain. Let $\hat{W}(s, \alpha, t)$ be the expected gain from the strategy $v^*(\cdot)$ found by solving equation (6.1), i.e.

$$\hat{W}(s, \alpha, t) = E^{s,\alpha,t}_{v^*(\cdot)} \Big[B(T) \Big] \ . \tag{6.3}$$

Let $W(s, \alpha, \tau = T - t) = \hat{W}(s, \alpha, t)$, and following the same steps as used to derive equation (6.1), (or simply setting $\lambda = 0$), we obtain

$$W_\tau \quad = \quad \mu s W_s + \frac{\sigma^2 s^2}{2} W_{ss} + e^{r\tau}\big(- v^* f(v^*)\big) s + g(v^*) s W_s + v^* W_\alpha \ . \tag{6.4}$$

The initial condition $W(s, \alpha, \tau = 0)$ is determined using the same arguments as used to derive (6.2)

$$W(s, \alpha, \tau = 0) = \lim_{v \to -\infty} \alpha s f(v). \tag{6.5}$$

[3] Note that while the initial point is changed from (s_1, α_1, t_1) to (s_2, α_2, t_2), the risk aversion level λ is kept constant.

6.3 Construction of the Efficient Frontier

For a given value of λ, we solve the nonlinear PDE (6.1), which gives us the optimal control $v^*(\cdot)$. With this optimal control $v^*(\cdot)$, we then solve the linear PDE (6.4). Let

$$\begin{aligned} V_0 &= V(s = S(0), \alpha = A(0), \tau = T) = E^{s,\alpha,t=0}_{v^*(\cdot)}\Big[B(T) - \lambda \int_0^T \big(A(t')dS(t')\big)^2\Big] , \\ W_0 &= W(s = S(0), \alpha = A(0), \tau = T) = E^{s,\alpha,t=0}_{v^*(\cdot)}\Big[B(T)\Big]. \end{aligned} \quad (6.6)$$

In order to produce a plot of reward (expected) versus risk, we define risk so that it has the same dimensions as the expected gain, i.e.

$$\text{Risk} = \left(E^{s,\alpha,t=0}_{v^*(\cdot)}\left[\int_0^T \big(A(t')dS(t')\big)^2\right]\right)^{1/2} = \sqrt{\frac{W_0 - V_0}{\lambda}} \ . \quad (6.7)$$

from equations (6.6). Equations (6.6) and (6.7) give us a single point on the efficient frontier. Repeating the above computation for different values of λ allows us to trace out the entire efficient frontier.

6.4 Localization and Boundary Conditions

6.4.1 Optimal Control Equation (6.1)

The original problem (6.1) is posed on the domain

$$(s, \alpha, \tau) \in [0, \infty] \times [0, \alpha_{init}] \times [0, T] \quad (6.8)$$

and we allow v to take any non-positive value. For computational purposes, we localize this domain to

$$\Omega = [0, s_{\max}] \times [0, \alpha_{init}] \times [0, T] \quad (6.9)$$

and impose $v \in [v_{\min}, 0]$ for some finite negative value $v_{\min}$.

At $\alpha = 0$, we do not permit selling which would cause $\alpha < 0$, therefore $v = 0$ and hence

$$V_\tau = \mu s V_S + \frac{\sigma^2 s^2}{2} V_{ss} \ ; \ \alpha = 0, \quad (6.10)$$

which does not require a boundary condition. Also, no boundary condition is required at $\alpha = \alpha_{init}$ since $v \le 0$.

At $s = 0$, no boundary condition is needed and we simply solve

$$V_\tau = \sup_{v \in [v_{\min}, 0]} \Big[v V_\alpha\Big] \ ; \ s = 0 \quad (6.11)$$

At $s = s_{\max}$, we make the assumption that $V \simeq C(\alpha, \tau) s^2$, which can be justified by noting that the term $\lambda \sigma^2 \alpha^2 s^2$ acts as a source term in equation (6.1). We also assume that the effect of any permanent price impact at $s = s_{\max}$ can be ignored i.e. $g(v) = 0$ at $s = s_{\max}$. This gives

$$V_\tau = (2\mu + \sigma^2)V - \lambda \sigma^2 \alpha^2 s^2 + \sup_{v \in [v_{\min}, 0]} \Big[e^{r\tau}(-v f(v))s + v V_\alpha\Big] \ ; \ s = s_{\max} \ . \quad (6.12)$$

Equation (6.12) is clearly an approximation. We will carry out numerical tests with varying $s_{\max}$ to show that the error in this approximation can be made small in regions of interest.

The initial condition at $\tau = 0$ is given by equation (6.2).

6.4.2 Expected Value Equation (6.4)

Similar to the situation for V, no boundary condition is needed for W at $\alpha = 0$, $\alpha = \alpha_{init}$ or $s = 0$. At $s = s_{\max}$, we assume $W \simeq D(\alpha, \tau)s$ (based on the initial condition (6.5)) and $g(v) = 0$. Consequently,

$$W_\tau = \mu W + e^{r\tau}(-v^* f(v^*))s + v^* W_\alpha \ ; \ s = s_{\max}. \tag{6.13}$$

Again, equation (6.13) is clearly an approximation. We will verify that the effect of this is small for sufficiently large $s_{\max}$.

The initial condition at $\tau = 0$ is given by equation (6.5).

7 HJB Equation Formulation: ABM case

The derivation is similar to that in the GBM case. In this section, we show that by assuming asset-price-independent temporary price impact (3.4) and zero interest rate, the optimal strategy has no s-dependence. Under these assumptions, a derivation similar to that in Appendix A gives the HJB PDE

$$V_\tau = \mu S(0) V_s + \frac{\sigma^2 S(0)^2}{2} V_{ss} - \lambda \sigma^2 \alpha^2 S(0)^2 + \sup_{v \le 0} \Big[v(V_\alpha - s) - v h(v) S(0) + g(v) S(0) V_s \Big] \ . \tag{7.1}$$

Note that the explicit s dependence in equation (7.1) appears only in the term $v(V_\alpha - s)$. Let

$$\hat{U}(s, \alpha, \tau) = V(\alpha, s, \tau) - \alpha s. \tag{7.2}$$

Substituting equation (7.2) into equation (7.1) gives

$$\hat{U}_\tau = \mu S(0)(\hat{U}_s + \alpha) + \frac{\sigma^2 S(0)^2}{2} \hat{U}_{ss} - \lambda \sigma^2 \alpha^2 S(0)^2 + \sup_{v \le 0} \Big[v \hat{U}_\alpha - v h(v) S(0) + g(v) S(0) \big(\hat{U}_s + \alpha\big) \Big] \ . \tag{7.3}$$

From equations (3.5) and (4.1), the initial condition for V is

$$V(s, \alpha, \tau = 0) = \lim_{v \to -\infty} \alpha(s + S(0) h(v)). \tag{7.4}$$

Therefore, from (7.2) we obtain

$$\hat{U}(s, \alpha, \tau = 0) = \lim_{v \to -\infty} \alpha(s + S(0) h(v)) - \alpha s = \lim_{v \to -\infty} \alpha S(0) h(v) \ . \tag{7.5}$$

Now, note that equation (7.3) has no explicit s dependence, and that the initial condition (7.5) has no s dependence. It therefore follows that equation (7.3) with initial condition (7.5) can be satisfied by a function

$$U(\alpha, \tau) = \hat{U}(s, \alpha, \tau) \tag{7.6}$$

where $U(\alpha, \tau)$ satisfies

$$U_\tau = \mu S(0) \alpha - \lambda \sigma^2 \alpha^2 S(0)^2 + \sup_{v \le 0} \Big[v U_\alpha - v h(v) S(0) + g(v) S(0) \alpha \Big] \ , \tag{7.7}$$

with

$$U(\alpha, \tau = 0) = \lim_{v \to -\infty} \alpha S(0) h(v) \ . \tag{7.8}$$

Proposition 1 *Assuming Arithmetic Brownian Motion (2.6), asset-price-independent temporary price impact (3.4), zero interest rate, and initial condition (7.4), the optimal control for equation (7.1) is static, even when optimization is over the class of dynamic strategies.*

Proof. The optimal control for equation (7.1) is same as the optimal control for equation (7.7), which is independent of s, i.e. $v^*(\cdot) : (\alpha, \tau) \mapsto v^*$, hence the optimal control for problem (7.1) is also independent of s, i.e. static. □

7.1 Special case analytical solution of (7.7)

In general, the PDE (7.7) has no known analytical solution. This section gives a special case analytical solution under the additional assumptions of zero drift, unconstrained control and linear price impact functions. More formally, we make the following set of common assumptions [4]

Assumption 7.1

$$\begin{aligned} dS(t) &= g(v)S(0)dt + \sigma S(0)d\mathbb{W}(t), \\ r &= 0, \\ h(v) &= \kappa_s \,\mathrm{sgn}(v) + \kappa_t v, \\ g(v) &= \kappa_p v, \\ v &\in (-\infty, \infty) \end{aligned} \tag{7.9}$$

which gives the following result.

Proposition 2 *Under Assumptions 7.1, the optimal control for (7.7) is identical with the (continuous equivalent of the) static strategy in Almgren and Chriss (2001); Almgren (2009), i.e.*

$$v^*(\alpha,\tau) = -\frac{\alpha K \cosh(K\tau)}{\sinh(K\tau)} \tag{7.10}$$

where $K = \sqrt{\lambda\,\sigma^2 S(0)/\kappa_t}$.

The value function $U(\alpha,\tau)$ *is*

$$U = E + \lambda F \ , \tag{7.11}$$

where

$$\begin{aligned} E &= \frac{S(0)\,\alpha\,\left(2\,\kappa_s f_1(\tau)^2 + \alpha\,\kappa_p f_1(\tau)^2 + \alpha\,\lambda\,\sigma^2 S(0)\,\tau + \alpha\,\kappa_t K f_1(\tau) f_2(\tau)\right)}{-2 f_1(\tau)^2}, \\ F &= \frac{\sigma^2 S(0)^2 \alpha^2 \left(-f_3(\tau)^2 f_1(\tau) - f_1(\tau) + 2\,\tau\, K f_3(\tau)\right)}{4K f_1(\tau)^2 f_3(\tau)} \ , \\ f_1(\tau) &= \sinh(K\tau) \ , \\ f_2(\tau) &= \cosh(K\tau) \ , \\ f_3(\tau) &= \exp(K\tau) \ . \end{aligned} \tag{7.12}$$

Note that if $\kappa_s = 0$*, then both* E *and* F*, and hence* U *are proportional to* α^2.

Proof. *Under Assumptions 7.1, the PDE (7.7) has the form*

$$U_\tau = -\lambda\sigma^2\alpha^2 S(0)^2 + \sup_{v\in(-\infty,\infty)} \left[vU_\alpha - \left(\kappa_s v\,\mathrm{sgn}(v) + \kappa_t v^2\right) S(0) + \kappa_p v S(0)\alpha \right] . \tag{7.13}$$

Using an initial condition that is consistent with (7.8) for $U(\alpha,\tau)$ *gives*

$$\begin{aligned} U(\alpha,0) &= 0 \quad ; \ \alpha = 0 \\ &= -\infty \ ; \ \text{otherwise} \ , \end{aligned} \tag{7.14}$$

by using the definitions (7.2) and (7.6). It can be verified by straightforward calculations that the value function (7.11) and the control (7.10) solves the HJB PDE (7.13), (7.14). □

In general, we would like to restrict v from taking all real values. For example, in the case of selling, a natural constraint is $v \leq 0$ (the default in this paper). This constraint may take effect if $\mu \neq 0$, in which case the analytical solution will no longer be valid.

[4] Note that the assumption of unconstrained control may not be desirable as it allows buying shares during stock liquidation.

8 Approximations to pre-commitment mean variance

As we mentioned in the introduction, the method typically used to evaluate the performance of an algorithmic execution strategy aligns well with the pre-commitment mean variance formulation. Here we give a brief but self-contained description of this formulation. We also discuss various approximations to this formulation.

8.1 Pre-commitment mean variance

For notational simplicity, we define $x = (s, b, \alpha)$ for a space state. Note that the state space is expanded to include b in this formulation, in contrast to the mean quadratic variation case. For a fixed initial point (x, t) where $t < T$, we define the functional

$$J^{MV}(x, t, v(\cdot); \lambda) \quad = \quad E^{x,t}_{v(\cdot)}\Big[B(T)\Big] - \lambda Var^{x,t}_{v(\cdot)}\Big[B(T)\Big] \ , \tag{8.1}$$

where $Var^{x,t}_{v(\cdot)}[\cdot]$ is the variance at the initial point (x, t) using the control $v(\cdot)$. Let $(x_0, 0) = (s_{init}, 0, \alpha_{init}, 0)$ be the initial state. The corresponding optimal strategy $v^*_{x_0,0,\lambda}(\cdot)$ is termed the pre-commitment mean variance optimal strategy (Basak and Chabakauri, 2010). We note that the optimal strategy $v^*_{x_0,0,\lambda}(\cdot)$ is dynamic, in both the GBM case (Forsyth, 2011) and the ABM case (Lorenz and Almgren, 2006; Lorenz, 2008).

The pre-commitment strategy is optimal in the following sense: suppose we carry out many thousands of trades. We then examine the post-trade data, and determine the realized mean return and the standard deviation. Assuming that the modeled dynamics very closely match the dynamics in the real world, the optimal pre-commitment strategy would result in the largest realized mean return, for a given standard deviation, compared to any other possible strategy.

Although the pre-commitment mean variance formulation is consistent with evaluation of performance of algorithmic trading trading strategies in practice, these optimal strategies are time-inconsistent (Basak and Chabakauri, 2010; Wang and Forsyth, 2010; Forsyth, 2011; Wang and Forsyth, 2011a), a property that is considered unnatural by some authors (Basak and Chabakauri, 2010).

8.1.1 Time-inconsistency of optimal strategies

Let (x_1, t_1) be some state at time t_1 and $v^*_{x_1,t_1,\lambda}(\cdot)$ be the corresponding optimal policy. Let (x_2, t_2) be some other state at time $t_2 > t_1$ and $v^*_{x_2,t_2,\lambda}(\cdot)$ be the corresponding optimal policy. We have time-inconsistency in the sense that

$$v^*_{x_1,t_1,\lambda}(x', t') \quad \neq \quad v^*_{x_2,t_2,\lambda}(x', t') \ ; \ t' \geq t_2 \quad . \tag{8.2}$$

As discussed in Basak and Chabakauri (2010), there is no direct dynamic programming principle for determining $v^*_{x_0,0,\lambda}(\cdot)$ due to time-inconsistency. Forsyth (2011) uses a Lagrange multiplier method to solve for $v^*_{x_0,0,\lambda}(\cdot)$.

We now discuss various approximations to the pre-commitment mean variance problem that lead to time-consistent optimal strategies.

8.2 Approximation 1: Restrict to static strategies

The Almgren and Chriss (2001) approximation essentially restricts the admissible strategies to static strategies in optimizing (8.1). As discussed in the previous section, this is suboptimal, in both the GBM and the ABM case. It is interesting that, for this approximation problem of maximizing the mean variance functional (8.1) *over static strategies* (assuming ABM), the optimal strategies are time-consistent. This time-consistency can be verified using the formula (7.10).

8.3 Approximation 2: Use quadratic variation as risk measure

Another approach is to approximate variance by quadratic variation, the accuracy of which is discussed in Appendix C. As discussed in Section 5.1, quadratic variation can be justified as a reasonable alternative risk measure. The current paper studies this approximation. Our more recent paper (Tse et al., 2011) compares optimal strategies in this approximation to the truly optimal strategies for the pre-commitment mean variance formulation. The time-consistency of this formulation is discussed in Section 5.3.

8.4 Approximation 3: Restrict to time-consistent strategies

If the only criticism of the pre-commitment mean variance formulation is that its optimal strategies are time-inconsistent, the optimization can be restricted to optimizing over time-consistent strategies. This is in some sense similar to Approximation 1, which optimizes over static strategies. However, the restriction to time-consistent strategies is more subtle, as we explain below.

The restriction to static strategies is easy to understand since we can look at a single strategy $v(\cdot)$ and determine whether it is static or not. This is not the case for the restriction to time-consistent strategies. It is important to note that time-consistency concerns the relation between a continuum of strategies, as explained in Section 5.3 and 8.1.1, and cannot be inferred from examination of a single strategy $v(\cdot)$.

Although it is more difficult to enforce time-consistency as a restriction (in terms of defining the corresponding class of admissible strategies), it turns out that this is not necessary to solve for the optimal time-consistent solutions. Essentially, since the time-consistent strategies follow the Bellman's principle of optimality by definition, the dynamic programming principle can be used to solve for the optimal time-consistent solutions. We refer readers to Basak and Chabakauri (2010) for details.

From a computational perspective, the optimal time-consistent strategies are in fact more difficult to determine (Wang and Forsyth, 2011b) compared to the pre-commitment optimal strategies. We also note that pre-commitment and time consistent strategies are the same as $T \to 0$ (Basak and Chabakauri, 2010).

In some special cases, the optimal time consistent strategies are identical to the optimal strategies in the mean-quadratic variation formulation (Bjork et al., 2009).

8.5 Connection between Approximation 1 and 2

Approximation 1 restricts attention to static strategies. Approximation 2 approximates variance by quadratic variation. These two approximations have the connection that variance is the same as (the expected value of) quadratic variation for static strategies, i.e.

$$Var_{v(\cdot)}^{x,t=0}\Big[B(T)\Big] = E_{v(\cdot)}^{x,t=0}\left[\int_0^T \big(A(t')dS(t')\big)^2\right] \tag{8.3}$$

under some additional mild assumptions detailed in Appendix C.

Nevertheless, we emphasize that this equality, by itself, does not imply that the static solution in Almgren and Chriss (2001) is optimal for the quadratic variation risk measure. The subtle point is that Almgren and Chriss (2001) does not study how dynamic strategies (i.e. use information regarding how the stock price evolves after the start of trading) perform in terms of the quadratic variation risk measure. In this paper, we provide a proof of this optimality. Recall that Section 7.1 shows that the strategy (7.10) solves (5.5), while Proposition 1 shows that no dynamic strategy is better than this static strategy.

In general, the equality (8.3) does not hold, and quadratic variation is only an approximation to variance. Although the accuracy of this approximation is good when trading impact is small (compared to volatility), this approximation can be poor when trading impact is larger but still realistic. Our proof in Appendix C shows precisely what is ignored in this approximation. Examples in which the approximation is poor can be found in Tse et al. (2011).

Remark 8.1 (Static as a restriction or as a property) *It should be clear at this point that it is important to distinguish between whether the use of the concepts static or dynamic refer to the class of admissible* *strategies, or to a property of the optimal control. In particular, even if the class of admissible controls is dynamic, the optimal control may turn out to be static. Proposition 1 is an example. Another example can be found in Schied et al. (2010).*

9 Numerical Method: GBM Case (6.1)

We give a brief outline of the numerical method used to solve equation (6.1). We will use a semi-Lagrangian method, similar to the approach used in Chen and Forsyth (2007).

Along the trajectory $s = s(\tau), \alpha = \alpha(\tau)$ defined by

$$\begin{aligned} \frac{ds}{d\tau} &= -g(v)s \\ \frac{d\alpha}{d\tau} &= -v \quad , \end{aligned} \tag{9.1}$$

equation (6.1) can be written as

$$\sup_{v \le 0} \left[\frac{DV}{D\tau}(v) - \mathcal{L}V - e^{r\tau}(-vf(v))s - \lambda \alpha^2 s^2 \sigma^2 \right] = 0 \quad , \tag{9.2}$$

where the operator $\mathcal{L}V$ is given by

$$\mathcal{L}V = \mu s V_s + \frac{\sigma^2 s^2}{2} V_{ss} \ , \tag{9.3}$$

and where the Lagrangian derivative $\frac{DV}{D\tau}(v)$ is given by

$$\frac{DV}{D\tau}(v) = V_\tau - V_s g(v) s - V_\alpha v \quad . \tag{9.4}$$

The Lagrangian derivative is the rate of change of V along the trajectory (9.1).

Define a set of nodes $[s_0, s_1, ..., s_{i_{max}}]$, $[\alpha_0, \alpha_1, ..., \alpha_{k_{max}}]$, discrete times $\tau^n = n\Delta\tau$, and localize the control candidates to values in a finite interval $[v_{\min}, v_{\max}]$. Let $V(s_i, \alpha_j, \tau^n)$ denote the exact solution to equation (6.1) at point (s_i, α_j, τ^n). Let $V_{i,j}^n$ denote the discrete approximation to the exact solution $V(s_i, \alpha_j, \tau^n)$.

We use standard finite difference methods (d'Halluin et al., 2005) to discretize the operator $\mathcal{L}V$ as given in (9.3). Let $(\mathcal{L}_h V)_{i,j}^n$ denote the discrete value of the differential operator (9.3) at node (s_i, α_j, τ^n). The operator (9.3) can be discretized using central, forward, or backward differencing in the s direction to give

$$(\mathcal{L}_h V)_{i,j}^n = a_i V_{i-1,j}^n + b_i V_{i+1,j}^n - (a_i + b_i) V_{i,j}^n \ , \tag{9.5}$$

where a_i and b_i are determined using the algorithm in d'Halluin et al. (2005).

Let $v_{i,j}^n$ denote the approximate value of the control variable v at mesh node (s_i, α_j, τ^n). Then we approximate $\frac{DV}{D\tau}(v)$ at $(s_i, \alpha_j, \tau^{n+1})$ by the following

$$\left(\frac{DV}{D\tau}(v) \right)_{i,j}^{n+1} \simeq \frac{1}{\Delta\tau} (V_{i,j}^{n+1} - V_{\hat{i},\hat{j}}^n) \tag{9.6}$$

where $V_{\hat{i},\hat{j}}^n$ is an approximation of $V(s_{\hat{i}}^n, \alpha_{\hat{j}}^n, \tau^n)$ obtained by linear interpolation of the discrete values $V_{i,j}^n$, with $(s_{\hat{i}}^n, \alpha_{\hat{j}}^n)$ given by solving equations (9.1) backwards in time for fixed $v_{i,j}^{n+1}$ to give

$$\begin{aligned} s_{\hat{i}}^n &= s_i + s_i g(v_{i,j}^{n+1}) \Delta\tau + O(\Delta\tau)^2 \\ \alpha_{\hat{j}}^n &= \alpha_j + v_{i,j}^{n+1} \Delta\tau \quad . \end{aligned} \tag{9.7}$$

Our final discretization is then

$$V_{i,j}^{n+1} = \sup_{v_{i,j}^{n+1} \in [v_{\min}, v_{\max}]} \left[V_{\hat{i},\hat{j}}^{n} + \Delta\tau \; e^{r\tau^{n+1}} (-v_{i,j}^{n+1} f(v_{i,j}^{n+1})) \right] + \Delta\tau (\mathcal{L}_h V)_{i,j}^{n+1} - \Delta\tau \lambda (\alpha_j)^2 s_i^2 \sigma^2 \; . \tag{9.8}$$

Let

$$\begin{aligned} \Delta s_{\max} &= \max_i s_{i+1} - s_i \\ \Delta \alpha_{\max} &= \max_j \alpha_{j+1} - \alpha_j \; , \end{aligned} \tag{9.9}$$

and define a discretization parameter h such that

$$h = \frac{\Delta\alpha_{\max}}{C_1} = \frac{\Delta s_{\max}}{C_2} = \frac{\Delta\tau}{C_3} \tag{9.10}$$

where C_i are positive constants. Note that we must solve a local optimization problem at each node at each time step in equation (9.8). In fact, we need to determine the global maximum of the local optimization problem. If the set of controls $[v_{\min}, v_{\max}]$ is discretized with spacing h, then a linear search of the control space will converge to the viscosity solution of the HJB equation (6.1) (Wang and Forsyth, 2008). An alternative (and less computationally expensive) method is to use a one dimensional optimization algorithm (Brent, 1973) to determine the local optimal control. The difficulty here is that one dimensional optimization methods are not guaranteed to converge to the global maximum. We will carry out tests using both methods in the following.

10 Numerical Method ABM Case (7.7)

Similar to the derivation in the last section, (7.7) can be written as

$$\sup_{v \le 0} \left[\frac{DU}{D\tau}(v) + v h(v) S(0) - g(v) S(0) \alpha \right] = \mu S(0)\alpha - \lambda \alpha^2 S(0)^2 \sigma^2 \tag{10.1}$$

where the Lagrangian derivative $\frac{DU}{D\tau}(v) = U_\tau - U_\alpha v$.

By integrating along the Lagrangian path and discretizing, we obtain

$$U_j^{n+1} = \sup_{v_j^{n+1} \in [v_{\min}, v_{\max}]} \left(U_{\hat{j}}^{n} + \Delta\tau \big(g(v_j^{n+1}) S(0) \alpha_j - v_j^{n+1} h(v_j^{n+1}) S(0) \big) \right) + \Delta\tau \left(\mu S(0) \alpha_j - \lambda \alpha_j^2 S(0)^2 \sigma^2 \right), \tag{10.2}$$

where we have used the notation $U_j^{n+1} = U(\alpha_j, \tau^{n+1})$, $v_j^{n+1} = v_j(\tau^{n+1})$ and $U_{\hat{j}}^n \approx U(\alpha_{\hat{j}}^n, \tau^n)$. Here $\alpha_{\hat{j}}^n$ is defined as in equation (9.7).

Either linear or quadratic interpolation can be used in approximating $U_{\hat{j}}^n$. Linear schemes have the advantage that they are monotone and numerical solutions are guaranteed to converge to the viscosity solution of the HJB equation, whereas quadratic schemes may not converge to the viscosity solution. In the special case where the analytical solution (7.12) is known, our quadratic interpolation scheme does converge to the exact solution of the HJB equation (10.1).

11 Numerical Examples

We solve both the GBM problem (6.1) and the ABM problem (7.1). Recall that the efficient frontier is constructed as described in Section 6.3, where we define

$$\text{Risk} = \left(E_{v^*(\cdot)}^{s,\alpha,t=0} \left[\int_0^T \big(A(t') dS(t') \big)^2 \right] \right)^{1/2} \tag{11.1}$$

Parameter	Value
σ	.40
T	1/12
μ	0.0
r	0.0
$S(0)$	100
$A(0)$	1.0
Action	Sell
κ_p	0.0
κ_t	.002
κ_s	0.0
β	1.0
v_{min}	$-10^5/T$
v_{max}	0.0
S_{max}	5000

Table 1: *Parameters for Case 1: selling an illiquid stock over a long trading horizon.*

Refinement	Time steps	S nodes	α nodes	v nodes
0	100	67	41	30
1	200	133	81	59
2	400	265	161	117
3	800	529	321	233
4	1600	1057	641	465

Table 2: *Grid and time step information for various levels of refinement for parametric case in Table 1.*

to have the same units as expected gain.

We will consider two cases. Case 1 considers an illiquid stock traded over a long time horizon (one month). We only consider the GBM model because the ABM model is unrealistic in this case. Case 2 considers a liquid stock traded over a short time horizon (one day). We consider both the GBM and the ABM model in this case and compare the results.

11.1 Example 1: Illiquid Stock, Long Trading Horizon (GBM)

The parameters for this case are shown in Table 1. The value of κ_t in Table 1 corresponds to a temporary price impact of about 240 bps for liquidating at a constant rate over the entire month. This would correspond to an illiquid stock. A relatively large volatility is also assumed, the value of σ in Table 1 corresponds to a standard deviation of about 1154 bps of $S(T)$.

11.1.1 Convergence Tests

We will first carry out some convergence tests, using the data in Table 1. The grid and time step information are given in Table 2

As noted in Section 9, in general, we need to use a linear search to guarantee that the global maximum of the local optimization problem at each node in equation (9.8) is determined to $O(h)$ for smooth functions. This guarantees convergence to the viscosity solution of equation (6.1).

Tables 3 and 4 compare results using a linear search or a one dimensional optimization technique to solve the local optimization problem at each node. These tables clearly show that both methods converge to the same solution. We have verified that the one dimensional optimization method converges to the

Refinement	Value Function	Expected Gain	Risk	Control
0	91.8439869	95.80303404	4.449183728	-40.5
1	91.9609793	95.85387412	4.411856106	-41.25
2	92.0205568	95.88081381	4.39332277	-41.625
3	92.0510158	95.88846915	4.38032725	-41.8125

TABLE 3: *Convergence test for using linear search of discrete trade rates for parametric case shown in Table 1. The grid and time step information for each refinement is given in Table 2. All values are reported at* $s = S(0) = 100, \alpha = A(0) = 1, \tau = T$ *for* $\lambda = 0.2$. *Value function is* V *as defined in (6.1). Expected gain is* W *as defined in (6.6). Risk is* $\sqrt{(W - V)/\lambda}$ *as defined in (6.7). Control is* v^* *as defined in (6.4), which is determined using a linear search. Compare with Table 5 which shows the result of using a one dimensional optimization method to determine* v^*.

Refinement	Value Function	Expected Gain	Risk	Control
0	91.8466222	95.80519281	4.448915945	-41.0986695
1	91.9616700	95.85496493	4.412082788	-41.4829134
2	92.0207373	95.88384474	4.394944527	-41.6657167
3	92.0510986	95.90026189	4.387005382	-41.7545063

TABLE 4: *Convergence test for using a one dimensional optimization method for parametric case shown in Table 1. Notations are as in Table 3 and results are computed in the same way except that the control is determined using a one dimensional optimization method. Note that the values in the two tables appear to converge to the same limit.*

global optimum in many tests, which we will not report. In the following, we will use the one dimensional optimization method, since it is much less expensive, computationally.

Recall that we made several approximations in order to determine boundary conditions at $S = S_{\max}$. Table 5 shows the effect of increasing $S_{\max}$, and verifies that the effect of these boundary condition approximations is negligible in regions of interest.

Another test of convergence is to consider the special case analytic solution for $\lambda = 0$. In this case where expected gain is maximized regardless of risk, the optimal selling strategy should sell at a constant rate to minimize temporary trading impact. This can be proved by noting the parametric choice $\mu = r = 0$ in Table 1 and the form of temporary price impact (3.3). Since $\mu = r = 0$, and the constant liquidation rate is $v = -1/T = -1/12$, the expected gain will be (using the parameters in Table 1)

$$E\Big[e^{-\kappa_t/T}\frac{1}{T}\int_0^T S(t)\ dt\Big] \simeq 97.6286 \tag{11.2}$$

Table 6 shows the results for the expected gains for $\lambda = 0.0001$. The table shows that the numerical results appear to be converging to the analytic solution for $\lambda = 0$.

Refinement	Value Function	Expected Gain	Risk	Control
0	91.8466222	95.80519281	4.448915945	-41.0986695
1	91.9616700	95.85496493	4.412082788	-41.4829134
2	92.0207373	95.88384474	4.394944527	-41.6657167

TABLE 5: *Test to confirm increasing* $S_{\max}$ *makes no difference for the parametric case as shown in Table 1. Notations are as in Table 4 and results are computed in the same way, except that this table uses* $S_{\max} = 20000$ *instead of* $S_{\max} = 5000$. *Note that there is negligible difference between the two tables.*

Refinement	Expected Gain
0	97.4841961
1	97.5447889
2	97.5807920
3	97.6017000
4	97.6136413

TABLE 6: *Convergence to analytical solution for constant liquidation rate for parametric case shown in Table 1. The grid and time step information for each refinement is given in Table 2. Expected gains W, as defined in (6.6), are reported at $s = S(0) = 100, \alpha = A(0) = 1, \tau = T$ for $\lambda = 0.0001 \approx 0$. Optimal control is determined using a one dimensional optimization method. Expected gains appear to converge to the analytical value* 97.6286.

11.1.2 Efficient Frontier

Figure 1 shows the efficient frontier traced out by the optimal trading strategies with different risk aversion level λ. Note that even the coarsest grid gives accurate results for expected gain values of interest.

11.1.3 Optimal Trading Rates

Figure 2 shows the optimal trading rate $v^*(s, \alpha, t; \lambda)$ as a function of asset price s. Note that the optimal strategy is slightly aggressive-in-the-money, i.e. the optimal strategy is to sell faster at larger s.

11.1.4 Comparison with pre-commitment mean-variance solution

The parametric case as shown in Table 1 is also studied in Forsyth (2011) which solves for the optimal strategies for the pre- commitment mean variance formulation. Here we briefly compare the two formulations from the perspective of numerical solution for the optimal controls. For more comprehensive comparison between the two formulations, we refer readers to (Tse et al., 2011).

It is more difficult to numerically solve for the optimal strategies in the pre-commitment mean variance formulation. Essentially this is because the local objective as a function of trading rate is sometimes very flat, making it difficult to determine the maximizer numerically. This flatness of the objective function is related to the fact that the variance risk measure does not concern the trading trajectory $A(t')$ but only the end result $B(T)$ as we discussed in Section 5.1. In contrast, this flatness is not observed in the mean quadratic variation formulation.

The numerical difficulties encountered in the pre-commitment mean variance formulation can be seen from two aspects. First, while both one dimensional optimization and linear search are able to find the maximizer in the mean quadratic variation formulation, as we discussed previously in this section, one dimensional optimization does not work well for the pre-commitment mean variance formulation and hence the more computationally expensive linear search method needs to be used. Second, in the pre-commitment mean variance formulation the optimal trading rate (as a function of asset price) is oscillatory, which reflects the near ill-posedness of this formulation (i.e. there are many strategies which give almost the same mean and variance). In contrast, the optimal trading rate is smooth in the mean quadratic variation formulation, as shown in Figure 2.

11.2 Example 2: Liquid Stock, Short Trading Horizon

The parameters for this case are shown in Table 7. The value of κ_t in Table 7 corresponds to a temporary price impact of about 5 bps for liquidating at a constant rate over the trading day. This would correspond to a liquid stock. The larger value of $\sigma = 1.0$ in Table 7 corresponds to a standard deviation of about 632 bps of $S(T)$. Therefore, this parametric case considers a situation where volatility is large compared to trading impact.

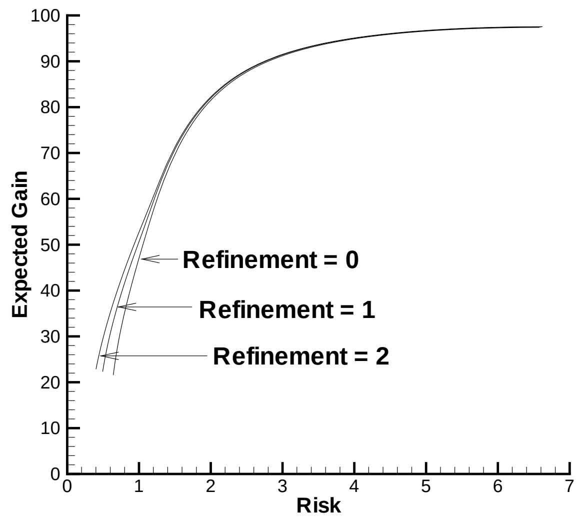

FIGURE 1: *The efficient frontier for parametric case shown in Table 1. The grid and time step information for each refinement is given in Table 2. Values of expected gain, as defined in (6.6), and risk, as defined in (6.7) are reported at* $s = S(0) = 100, \alpha = A(0) = 1, \tau = T$ *for various risk aversion level* λ. *Smaller values of* λ *represent less risk-averse strategies which have larger risks and expected gains.*

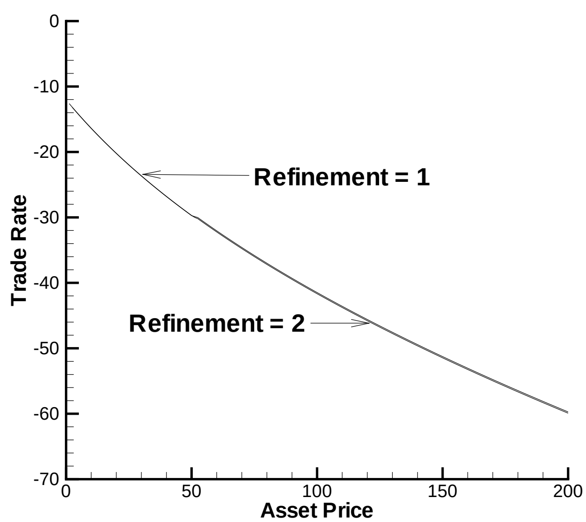

FIGURE 2: *Optimal trading rate* $v^*(s, \alpha, t; \lambda)$ *as a function of* s *at* $t = 0$, $\alpha = 1$, *and* $\lambda = 0.2$. *The risk aversion level* $\lambda = 0.2$ *corresponds to the point on the efficient frontier in Figure 1 with expected gain* 95.9 *and risk* 4.4. *Compare the trading rates with the constant liquidation rate* $v = -12$. *This is for the parametric case shown in Table 1. The grid and time step information for each refinement is given in Table 2.*

Parameter	Value
σ	1.0
T	1/250
κ_t	2×10^{-6}

Table 7: *Parameters for Case 2: selling a liquid stock over a short trading horizon. Other parameters are as given in Table 1.*

Refinement	Time steps	S nodes	α nodes	v nodes
0	800	67	41	30
1	1600	133	81	59
2	3200	265	161	117
3	6400	529	321	233

Table 8: *Grid and time step information for various levels of refinement for parametric case in Table 7.*

For both the GBM case and the ABM case, the grid and time step information are given in Table 8. Recall that in the GBM case, we solve equation (6.1); in the ABM case, we solve equation (7.7).

11.2.1 Geometric Brownian Motion

The efficient frontier for the GBM case is shown in Figure 3. Note that even the coarsest grid gives accurate results for expected gain values of interest, similar to the parametric case in illiquid stock, long trading horizon case in Table 1. Convergence results are shown in Table 9 for various values of λ. We note that convergence appears to be at a first order rate. The optimal trading rates as a function of asset price are shown in Figure 3 for various values of λ. It shows that the optimal strategies are more aggressive-in-the-money (slope of the curves are larger) for larger values of λ.

11.2.2 Arithmetic Brownian Motion

The ABM case also uses the parameters in Table 7 with the form of temporary pricing impact changed to (3.5). Convergence results are shown in Table 10. Note that the numerical values appear to converge to the analytical solution. Optimal trading rates are not plotted for the ABM since they are independent of the asset price s and can be obtained in Table 10.

11.2.3 Using optimal strategies from ABM as approximate solutions for GBM dynamics

Section 11.2.1 assumes the stock price process follows GBM and solves for the optimal strategies, which are dynamic. Section 11.2.2 assumes the stock price process follows ABM and solves for the optimal strategies, which turn out to be static. In this section, we compare the performance of the strategies in these two cases, assuming the stock price process follows GBM and the temporary price impact is of the form (3.2), as in Section 11.2.1. Note that by making these assumptions, the GBM strategies are truly optimal whereas the ABM strategies are not. The reason for conducting this comparison is that the static strategies in 11.2.2 have analytical solutions which can be considered as easy-to-compute approximate solutions to the optimal dynamic strategies in 11.2.1.

This comparison is shown in Figure 5 where we compare the efficient frontiers obtained by the truly optimal dynamic strategies and the approximate static strategies. In Figure 5, the frontier labeled with "Exact Control" is the same as in Figure 3, the frontier labeled with "Approximate Control" is generated using the static strategy approximation. Surprisingly, the frontier generated by the approximate solution is virtually identical with the truly optimal one. This indicates that there is essentially no error, as far as

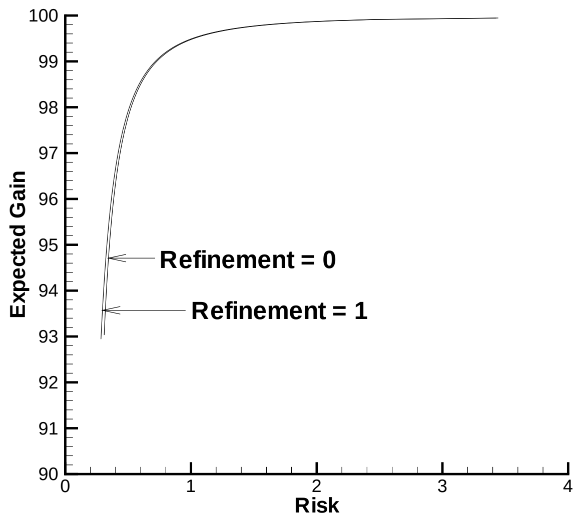

FIGURE 3: *The efficient frontier for parametric case shown in Table 7 in the GBM case. The grid and time step information for each refinement is given in Table 8. Values of expected gain and risk are reported at* $s = S(0) = 100, \alpha = A(0) = 1, \tau = T$ *for various risk aversion level* λ*. Smaller values of* λ *represent less risk-averse strategies which have larger risks and expected gains.*

Refinement	Expected Gain	Risk	Control
	$\lambda = 100$		
1	92.942727	.284003	-69312.3
2	92.926803	.271555	-72483.6
3	92.925507	.265444	-74491.4
	$\lambda = 10$		
1	97.756612	.482818	-22205.4
2	97.602898	.476058	-22552.9
3	97.620839	.472658	-22718.4
	$\lambda = 1$		
1	99.287578	.847254	-7058.17
2	99.290223	.843065	-7090.36
3	99.291576	.840960	-7106.59
	$\lambda = .2$		
1	99.68109	1.26727	-3140.72
2	99.68241	1.26168	-3156.80
3	99.68308	1.25887	-3164.78

TABLE 9: *Convergence test for using one dimensional optimization for parametric case shown in Table 7 in the GBM case. The grid and time step information for each refinement is given in Table 8. Notations are as in Table 3. All values are reported at* $s = S(0) = 100, \alpha = A(0) = 1, \tau = T$ *for various values of* λ*. Note that convergence appears to be at first order rate. Compare the trading rates with the constant liquidation rate* $v = -250$*.*

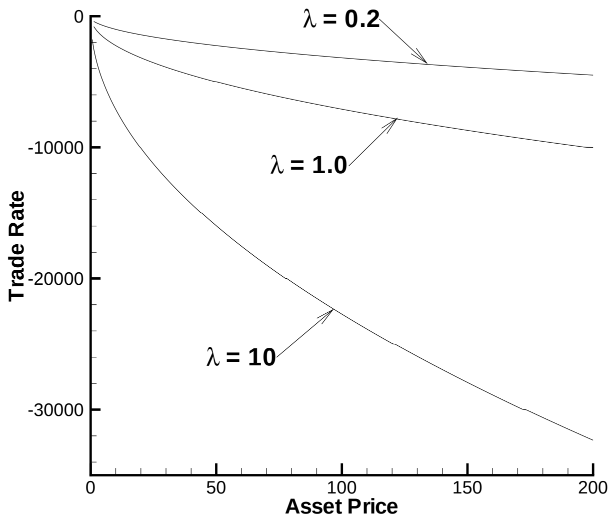

FIGURE 4: *Optimal trading rate $v^*(s, \alpha, t; \lambda)$ as a function of s at $t = 0$ and $\alpha = 1$ for various values of λ. This is for the parametric case shown in Table 7 in the GBM case. The grid and time step information for each refinement is given in Table 8. Compare the trading rates with the constant liquidation rate $v = -250$.*

Refinement	Expected Gain	Risk	Control
$\lambda = 100$			
1	92.336427	.304879	-70000.0
2	92.612649	.284365	-70341.5
3	92.770613	.274943	-70588.9
analytic	92.928932	.265915	-70710.7
$\lambda = 10$			
1	97.694545	.493519	-22312.0
2	97.728951	.483026	-22352.6
3	97.746400	.477908	-22359.0
analytic	97.763932	.472871	-22360.7
$\lambda = 1$			
1	99.281307	.853065	-7070.30
2	99.287096	.846951	-7070.88
3	99.289994	.843916	-7071.02
analytic	99.292893	.840896	-7071.07
$\lambda = .2$			
1	99.678547	1.267777	-3174.84
2	99.681165	1.262605	-3168.52
3	99.682470	1.260019	-3165.41
analytic	99.683772	1.257433	-3162.28

TABLE 10: *Convergence test for parametric case shown in Table 7 in the ABM case. The grid and time step information for each refinement is given in Table 8. Notations are as in Table 3. All values are reported at $s = S(0) = 100, \alpha = A(0) = 1, \tau = T$ for various values of λ. Note that the numerical results appear to converge to the analytical solution. Compare the trading rates with the constant liquidation rate $v = -250$.*

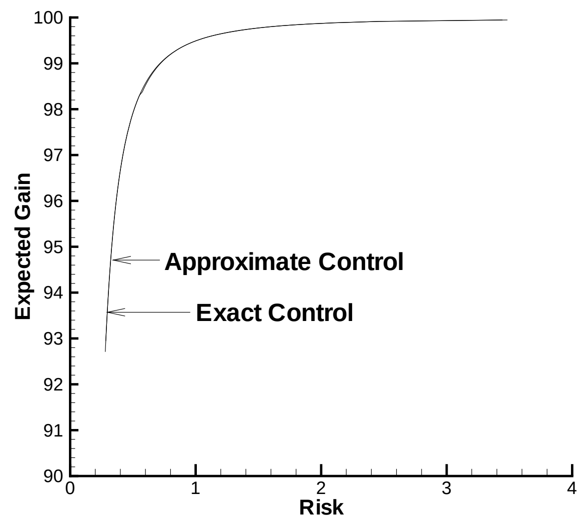

FIGURE 5: *Comparison of the efficient frontier computed by solving equation (6.1) (exact control) compared to constructing the efficient frontier using the control from the ABM approximation (7.7) (approximate control). The asset price process follows GBM and the temporary price impact takes the form (3.2) in computing both efficient frontiers.*

the efficient frontier is concerned, even though the static strategies trade at a different rate than the truly optimal dynamic strategies trade.

The accuracy of the static ABM strategies in approximating the dynamic GBM strategies can be explained as follows. At higher level of expected gains in Figure 5, both strategies must sell near the constant liquidation rate (throughout the entire trading horizon) to reduce trading impact, and therefore are similar. At lower level of expected gains, both strategies sell very quickly near $t = 0$, and actually finish most of the liquidation before the share price has time to move significently, i.e. most liquidation happens at $S(t) \approx S(0) = 100$. Note that for the same level of expected gain, the two strategies trade at similar rates[5] for s near 100, which implies that they trade at similar rates near $t = 0$. Consequently, the ABM approximation is also accurate for lower level of expected gain.

The observation above suggests that by increasing only volatility (to values much larger than $\sigma = 1$ considered here), the ABM approximation will still give efficient frontiers that are close to optimal for all risk levels. However, if volatility and temporary trading impacts are both increased substantially (to unrealistic levels), then we will see the sub-optimality of the ABM approximation at lower level of expected gains, though it is still accurate at higher level of expected gains. These conjectures are indeed confirmed by our numerical results, which are not reported here due to the unrealistic parameter values.

The above discussion may give the wrong impression that it suffices to have the correct trading rate at $s = 100$ and being aggressive-in-the-money has no advantage. We emphasize, however, that this an over-simplification that is not true in general. When risk is measured by variance, instead of quadratic variation, being aggressive in the money can reduce risk substantially (Tse et al., 2011). In other words, the above simplification serendipitously happens to work when risk is measured by quadratic variation. This illustrates that quadratic variation and variance are different risk measures that lead to different optimal strategies, as we point out in Section 5.1 and elaborated in (Tse et al., 2011).

[5]However, the difference increases as s moves away from 100. See Figure 4.

12 Conclusion

We have proposed a mean–quadratic-variation objective function for determining the optimal trade execution strategy. Quadratic variation as a risk measure takes account of the entire trading trajectory. This is in contrast with using variance as a risk measure which only considers the terminal portfolio value distribution. The static strategy in Almgren and Chriss (2001), which is originally derived as an approximate solution to the pre-commitment mean variance problem, turns out to be the truly optimal solution in the mean quadratic variation formulation (assuming ABM).

We have developed numerical schemes for solution of the mean–quadratic-variation optimal control problem, assuming either GBM or ABM. Any type of constraint can be imposed on the trading strategy. For example, the natural constraint when selling is that no intermediate buying is allowed.

In the GBM case, the optimal strategy depends smoothly on the underlying asset price. Numerical difficulties seen in the pre-commitment mean-variance formulation (Forsyth, 2011) are not seen in the mean quadratic variation formulation. In the ABM case, the optimal strategies are static, and thus different from those in the GBM case. Surprisingly, it turns out that the static strategies can be used as excellent approximations for the GBM case, even when volatility is large. We note that this accuracy of the static approximation does not hold in the pre-commitment mean variance formulation.

Finally, we emphasize that in general, mean quadratic variation and mean variance are not the same objective functions, and that the optimal strategies in each case can be significantly different. However, there are arguments to made for choosing each of these objective functions.

A Optimal Control

In this appendix, we give the steps used to derive equation (6.1). Using equation (2.5), then the risk term becomes

$$\int_t^T \left(A(t')\ dS(t')\right)^2 = \int_t^T \sigma^2 A(t')^2 S(t')^2\ dt'\ , \tag{A.1}$$

so that by equations (5.5) and (5.3) we have (using $E_{v(\cdot)}^{s,\alpha,t}\left[E_{v(\cdot)}^{s+\Delta s,\alpha+\Delta\alpha,t+\Delta t}[\cdot]\right] = E_{v(\cdot)}^{s,\alpha,t}[\cdot]$)

$$\begin{aligned}
\hat{V}(s,\alpha,t;\lambda) &= \sup_{v(\cdot)} E_{v(\cdot)}^{s,\alpha,t}\left[B(T) - \lambda \int_t^T \left(A(t')\ dS(t')\right)^2\right] \\
&= \sup_{v(\cdot)} E_{v(\cdot)}^{s,\alpha,t}\left[\int_t^T \left[e^{r(T-t')}\big(-vS_{exec}(v,t')\big) - \lambda\sigma^2 A(t')^2 S(t')^2\right] dt' + \lim_{v\to-\infty} A(T^-)S_{exec}(v,T^-)\right] \\
&= \sup_{v(\cdot)} E_{v(\cdot)}^{s,\alpha,t}\left[\int_t^{t+\Delta t} \left[e^{r(T-t')}\big(-vS_{exec}(v,t')\big) - \lambda\sigma^2 A(t')^2 S(t')^2\right] dt' \right.\\
&\qquad + E_{v(\cdot)}^{s+\Delta s,\alpha+\Delta\alpha,t+\Delta t}\int_{t+\Delta t}^T \left[e^{r(T-t')}\big(-vS_{exec}(v,t')\big) - \lambda\sigma^2 A(t')^2 S(t')^2\right] dt' \\
&\qquad \left. + E_{v(\cdot)}^{s+\Delta s,\alpha+\Delta\alpha,t+\Delta t}\left[\lim_{v\to-\infty} A(T^-)S_{exec}(v,T^-)\right]\right]
\end{aligned} \tag{A.2}$$

Noting that for any control $v(\cdot) : (S(t'), A(t'), t') \mapsto v,\ t' \geq t + \Delta t$,

$$\begin{aligned}
&E_{v(\cdot)}^{s+\Delta s,\alpha+\Delta\alpha,t+\Delta t}\left[\int_{t+\Delta t}^T \left[e^{r(T-t')}\big(-vS_{exec}(v,t')\big) - \lambda\sigma^2 A(t')^2 S(t')^2\right] dt' + \lim_{v\to-\infty} A(T^-)S_{exec}(v,T^-)\right] \\
&= J(s+\Delta s, \alpha+\Delta\alpha, t+\Delta t, v(\cdot);\lambda) \\
&\leq \sup_{v(\cdot)} J(s+\Delta s, \alpha+\Delta\alpha, t+\Delta t, v(\cdot);\lambda) = \hat{V}(s+\Delta s, \alpha+\Delta\alpha, t+\Delta t;\lambda).
\end{aligned} \tag{A.3}$$

with equality in the case of the optimal control $v^*(\cdot)$.

From equations (A.2-A.3) and the form of the price impact (3.2),

$$\hat{V}(s,\alpha,t;\lambda) = \sup_{v(\cdot)} E^{s,\alpha,t}_{v(\cdot)}\Big[e^{r(T-t)}(-vf(v)s)\Delta t - \lambda\sigma^2\alpha^2 s^2\Delta t + \hat{V}(s+\Delta s,\alpha+\Delta\alpha,t+\Delta t;\lambda)\Big] + O((\Delta t)^2). \quad \text{(A.4)}$$

Defining

$$\Delta\hat{V} = \hat{V}(s+\Delta s,\alpha+\Delta\alpha,t+\Delta t;\lambda) - \hat{V}(s,\alpha,t;\lambda), \quad \text{(A.5)}$$

and rearranging equation (A.4) gives

$$0 = \sup_{v(\cdot)} E^{s,\alpha,t}_{v(\cdot)}\Big[e^{r(T-t)}(-vf(v)s)\Delta t - \lambda\sigma^2\alpha^2 s^2\Delta t + \Delta\hat{V}\Big] + O((\Delta t)^2)\ . \quad \text{(A.6)}$$

From equations (2.3) and (2.5), using Ito's Lemma we obtain

$$E^{s,\alpha,t}_{v(\cdot)}[\Delta\hat{V}] = \Delta t\Big[\hat{V}_t + (\mu + g(v)s\hat{V}_s + \frac{\sigma^2 s^2}{2}\hat{V}_{ss} + v\hat{V}_\alpha\Big] + O((\Delta t)^{3/2})\ . \quad \text{(A.7)}$$

Let $V = V(s,\alpha,\tau = T-t;\lambda) = \hat{V}(s,\alpha,t;\lambda)$. Substituting equation (A.7) into equation (A.6), dividing by Δt, and letting $\Delta t \to 0$, then we obtain the HJB PDE for $\tau > 0$

$$V_\tau = \mu s V_s + \frac{\sigma^2 s^2}{2}V_{ss} - \lambda\sigma^2\alpha^2 s^2 + \sup_v\Big[e^{r\tau}(-vf(v))s + g(v)sV_s + vV_\alpha\Big]\ . \quad \text{(A.8)}$$

B Form of Permanent Price Impact (3.1)

Since temporary impact always leads to trading losses, there is no restriction on the functional form of the temporary impact (Huberman and Stanzl, 2004; Almgren et al., 2004). In contrast, the form of the permanent price impact must be restricted to ensure no-arbitrage, as noted by Huberman and Stanzl (2004). In this Appendix, we show that a permanent price impact function of the form (3.1) is consistent with the no-arbitrage condition of Huberman and Stanzl (2004), which basically states that the expected gain from a round trip trading strategy should be non-positive.

Note that while previous work considered only Arithmetic Brownian Motion, we handle both cases here. Since the proofs are very similar, we only give full details for the GBM case here.

In the following, we assume that there is no temporary price impact, since this is irrelevant in terms of no-arbitrage. Further, we assume that the deterministic drift term $\mu = 0$ in equation (2.5), and $g(v(t)) = \kappa_p v(t)$, with $\kappa_p = constant$.

Consequently, we consider a process of the form

$$dS(t) = \kappa_p v(t)S(t)dt + \sigma S(t)d\mathbb{W}(t)\ . \quad \text{(B.1)}$$

The solution of this SDE is

$$S(T) = S(0)\exp\Big[\kappa_p\int_0^T v(t)\ dt\Big]\exp\Big[\sigma\mathbb{W}(T) - \sigma^2 T/2\Big]\ . \quad \text{(B.2)}$$

Noting that

$$\int_0^T v(t)\ dt = \int_0^T \frac{dA(t)}{dt}dt = \int_0^T dA(t) = A(T) - A(0) \quad \text{(B.3)}$$

then equation (B.2) becomes

$$S(T) \quad = \quad S(0)\exp\Big[\kappa_p(A(T)-A(0))\Big]\exp\Big[\sigma\mathbb{W}(T)-\sigma^2T/2\Big] \ , \tag{B.4}$$

and consequently

$$E[S(T)] \quad = \quad S(0)\exp\Big[\kappa_p(A(T)-A(0))\Big] \ . \tag{B.5}$$

For a round trip trade, $A(T) = A(0)$, hence

$$E[S(T)] \quad = \quad S(0) \ . \tag{B.6}$$

Let $R(t)$ be the revenue from a trading strategy $v(t)$, so that

$$dR(t) \quad = \quad -v(t)S(t)\ dt. \tag{B.7}$$

Rearranging equation (B.1), we obtain

$$v(t)S(t)\ dt \quad = \quad \frac{dS(t)}{\kappa_p} - \frac{\sigma S(t)}{\kappa_p}\ d\mathbb{W}(t) \ . \tag{B.8}$$

Substituting equation (B.8) into (B.7) gives

$$R(T) = -\int_0^T \left[\frac{dS(t)}{\kappa_p} - \frac{\sigma S(t)}{\kappa_p}\ d\mathbb{W}(t)\right] = -\frac{S(T)-S(0)}{\kappa_p} + \frac{\sigma}{\kappa_p}\int_0^T S(t)\ d\mathbb{W}(t) \ . \tag{B.9}$$

Noting that

$$E\Big[\int_0^T S(t)\ d\mathbb{W}(t)\Big] = 0 \ , \tag{B.10}$$

then, for a round trip trade (from equation (B.6))

$$E[R(T)] = -\frac{E[S(T)]-S(0)}{\kappa_p} = 0 \ . \tag{B.11}$$

Consequently, the expected revenue for any round trip trade for a permanent price impact of the form (3.1) is zero, hence this precludes arbitrage. Note that equation (B.6) also holds in the ABM case, and the rest of the proof is similar.

C Derivation of Equation (8.3)

In this Appendix, we reconstruct the arguments used to derive equation (8.3). The reader should note the following assumptions:

AS1 The underlying process $S(t)$ has no drift and

$$dS(t') \quad = \quad \sigma(S(t'),t')\ d\mathbb{W}(t') \ . \tag{C.1}$$

AS2 The control $v(\cdot)$ is of the form $v(\cdot) : (S(t), A(t), t) \mapsto v$.

AS3 $r = 0$, $A(T^-) = A(T) = 0$, and the temporary impact is of the form (3.4)

From equations (2.4), (3.4) we have that

$$dB(t') = -vS(t')dt' - S(0)vh(v)dt' = -S(t')dA(t') - S(0)vh(v)dt', \tag{C.2}$$

Using the integration by part formula for stochastic integrals on the product $A(t')S(t')$, we have

$$-S(t')dA(t') = -d(S(t')A(t')) + A(t')dS(t') \tag{C.3}$$

since $dA(t')dS(t') = o(dt')$. Consequently,

$$dB(t') = -d(S(t')A(t')) + A(t')dS(t') - S(0)vh(v)dt'. \tag{C.4}$$

Integrating (C.4) from 0 to T, and noting $A(T^-) = A(T) = 0$, gives

$$B(T) = S(0)A(0) + \int_0^T A(t')dS(t') - \int_0^T S(0)vh(v)dt' \tag{C.5}$$

Note that the last term $B_{impact} \equiv \int_0^T S(0)vh(v)dt' = \int_0^T S(0)v(S(t'), A(t'), t')h(v(S(t'), A(t'), t'))dt'$ corresponds to the cost from nonzero trading impact and is stochastic in general. Now we make the assumption

AS2* The control $v(\cdot)$ is of the form $v(\cdot) : (A(t), t) \mapsto v$, i.e. a static strategy that is independent of $S(t)$.

With this assumption, the term $B_{impact} = \int_0^T S(0)v(A(t'), t')h(v(A(t'), t'))dt'$ becomes deterministic. As a result, equation (C.5) implies

$$Var\Big[B(T)\Big] = Var\Big[\int_0^T A(t')dS(t')\Big] = E\Big[\Big(\int_0^T A(t')dS(t')\Big)^2\Big] = E\Big[\Big(\int_0^T A(t')\sigma(S(t'), t')d\mathbb{W}(t')\Big)^2\Big] \tag{C.6}$$

since the Ito integral has zero expectation. Now we have

$$E\Big[\Big(\int_0^T A(t')\sigma(S(t'), t')\ d\mathbb{W}(t')\Big)^2\Big] = E\Big[\int_0^T \big(A(t')\sigma(S(t'), t')\big)^2 dt'\Big] = E\Big[\Big(\int_0^T A(t')dS(t')\Big)^2\Big], \tag{C.7}$$

where the first equality is a result of the Ito isometry.

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