Convergence of the Embedded Mean-Variance Optimal Points With Discrete Sampling*

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Abstract

A numerical technique based on the embedding technique proposed in [21, 33] for dynamic mean-variance (MV) optimization problems may yield spurious points, i.e. points which are not on the efficient frontier. In [27], it is shown that spurious points can be eliminated by examining the left upper convex hull of the solution of the embedded problem. However, any numerical algorithm will generate only a discrete sampling of the solution set of the embedded problem. In this paper, we formally establish that, under mild assumptions, every limit point of a suitably defined sequence of upper convex hulls of the sampled solution of the embedded problem is on the original MV efficient frontier. For illustration, we discuss an MV asset-liability problem under jump diffusions, which is solved using a numerical Hamilton-Jacobi-Bellman partial differential equation approach.

Keywords: mean-variance, scalarization optimization, embedding, Pareto optimal, asset-liability, Hamilton-Jacobi-Bellman (HJB) equation, jump diffusion

AMS Classification: 65K29, 91G60, 93C20

1 Introduction

The main objective of this paper is to analyze convergence properties of the computed mean-variance (MV) scalarization optimal points, sequenced by the embedding parameter sampling level, in the embedding technique for multi-period MV optimization.

1.1 Motivation

Many optimal stochastic control problems in finance can be formulated as a multi-period or continuous time MV optimization problem. Typical examples include portfolio optimization [7, 21, 25, 28, 29, 33], asset-liability management [5, 10, 15, 19, 20, 31], and optimal trade execution[17, 23].

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In this approach, we seek the optimal trade-off between the two conflicting criteria of maximizing the expected wealth of the investment (or trading), over a given time horizon, and minimizing investment risk. More specifically, letting W_t denote the total wealth from the investment at time t, we aim to maximize $\mathcal{E} = E[W_T]$ and minimize $\mathcal{V} = Var[W_T]$, where T is the end of the investment/trading horizon. Here, $E[\cdot]$ and $Var[\cdot]$ respectively denote the expectation and the variance operators.

Mean-variance optimization typically yields pre-commitment strategies [4, 8], which are not time-consistent [28, 29, 30]. There has been much discussion about such strategies in the economics literature [8]. However, it is argued in [28] that pre-commitment strategies are appropriate in the context of pension plan investment. It has also been pointed out that, in the context of optimal trade execution, the pre-commitment strategy optimizes trading efficiency as measured in practice [1]. The pre-commitment policy has also been commonly applied in insurance applications [9, 15, 19, 32].

As an illustration to relevant issues addressed in this paper, we consider the following application. Consider an investor who has a fixed initial wealth, which can be invested in (i) a risk-free asset, e.g., a government bond, or (ii) a risky asset, e.g., a stock market index. We assume that the investor can dynamically transfer wealth between these two assets. In addition to these assets, we assume that the investor also has fixed liabilities, in the form of deterministic cash outflows. These cash outflows are withdrawn at a set of pre-determined (event) dates. These cash outflows are usually specified in terms of an initial withdrawal and subsequent withdrawals equal to the initial withdrawal inflated at a known inflation rate. This asset-liability problem is assumed to continue over a relatively long horizon, e.g. 20 years.

The problem described above can be viewed as a prototype for the asset allocation problem, faced by the holder of a defined contribution pension plan (DCPP) in the sense that, upon retirement, the holder of a DCPP must invest his assets to generate living expenses over a long term horizon. Most existing literature for DCPP adopts an utility function based approach, see e.g., [28] and references therein. This may be partly due to the fact that it is more challenging to numerically determine the dynamic investment strategy which is optimal in the MV sense. As another concrete example, we can consider the case of a charitable endowment, where fixed cash flows (i.e. staff salaries) must be funded by an endowment which is invested in risky assets.

In both cases, the investment strategy can be modeled as a fraction of the total wealth invested in the risky asset. In the example considered in this paper, we assume that the underlying risky asset follows a jump diffusion process and we constrain the leverage ratio. To the best of our knowledge, no closed-form solutions for this problem are presently available in the literature.

1.2 Background

Following a standard scalarization method for multi-criteria optimization, a single criterion can be formed by a positively weighted sum of the criteria. Unfortunately, in the case of MV optimization, dynamic programming is not directly applicable to the resulting single-objective optimization problem, due to the presence of the variance term $Var[W_T]$.

1.2.1 Embedding Approach

To overcome this difficulty, a technique is proposed in [21, 33] to embed the objective of the MV scalarization problem in a new optimization problem, which involves $\mathcal{E} = E[W_T]$, $\mathcal{Q} = E[W_T^2]$, and an embedding parameter $\gamma \in (-\infty, +\infty)$, instead of $\mathcal{E}$, $\mathcal{V}$, and a positive scalarization parameter

$\mu > 0$. The dynamic programming principle can be applied to the embedded optimization problem, which gives rise to a non-linear Hamilton-Jacobi-Bellman (HJB) equation, from which optimal solutions with respect to the embedded problem can be obtained. For each embedding parameter γ, a pair $(\mathcal{E}, \mathcal{V})$ of values is associated with a solution to the corresponding HJB equation, see, e.g., [13, 17, 29].

We denote by $\mathcal{Y}_P$ the set of all $(\mathcal{V}, \mathcal{E})$ corresponding to the original MV scalarization problem. We will also refer to $\mathcal{Y}_P$ as the set of scalerization optimal points (SOPs) w.r.t. $\mathcal{Y}$. Let $\mathcal{Y}_Q$ be the set of all achievable $(\mathcal{V}, \mathcal{E})$ whose combination using an embedding parameter γ yields the optimal value of the embedded problem with the embedding parameter γ. Our goal is to determine the set $\mathcal{Y}_P$ numerically. It has been established in [21, 33] that the original MV scalarization optimal set $\mathcal{Y}_P$ is a subset of the embedded MV objective set $\mathcal{Y}_Q$. This result has led to widespread adoption of the embedding technique in MV optimization.

Unfortunately, the result that $\mathcal{Y}_P \subset \mathcal{Y}_Q$ is insufficient by itself, since there may exist *spurious* points, i.e., points in $\mathcal{Y}_Q$ but not in $\mathcal{Y}_P$. This problem can arise from nonconvexity of the original problem. Furthermore a point in $\mathcal{Y}_Q$ is a point in $\mathcal{Y}_P$ only for an embedding parameter satisfying necessary conditions. It is however difficult to verify these conditions in numerical computation; consequently a method for eliminating spurious points is required. Note that the imposition of the necessary conditions is not an issue when closed form solutions are available since the necessary conditions can be imposed explicitly (e.g. see [33]).

The issue of potential spurious points for the embedding method in the context of numerical computation was discussed in [27]. This raises an important issue of how to develop an algorithm for elimininating these points. This issue is partially addressed in [27] by identification of scalarization optimal points with respect to the embedded MV objective set. Specifically, it is shown that a spurious point is a point at which a supporting hyperplane for the embedded MV objective set does not exist, i.e. non-SOPs.

It is further noted in [27] that the full embedded objective set is not available in computation, since any numerical algorithm can compute only a single MV point corresponding to a given embedding γ. As a result, the requirement of constructing the full embedded MV set is relaxed, and the focus is on the computed objective set [27]. The *computed* embedded MV objective set $\mathcal{Y}_Q^\dagger$ is defined as the embedded MV objective set with a single embedded MV objective point for each embedding parameter. It is shown in [27] that the spurious points are non-SOPs with respect to the computed MV set $\mathcal{Y}_Q^\dagger$. These theoretical results yield a post-processing technique for the embedding method. This technique is applied to remove spurious points, which are now points in $\mathcal{Y}_Q^\dagger$ but not in $\mathcal{Y}_P$. This requires verification of the existence of a supporting hyperplane at each point in the set $\mathcal{Y}_Q^\dagger$, and hence, has a simple geometrical interpretation. We denote by $\mathcal{S}(\mathcal{Y}_Q^\dagger)$ the set of points in $\mathcal{Y}_Q^\dagger$ at which supporting hyperplanes exist. The main result in [27] is that $\mathcal{S}(\mathcal{Y}_Q^\dagger) = \mathcal{Y}_P$.

1.2.2 Alternative Approaches

There are several other techniques which can be used to circumvent the problem due to the variance term in MV optimization. A Martingale method, which is based on the use of Backward Stochastic Differential Equations (BSDEs) was used in [7]. Another method, also using BSDEs, is described in [12]. This technique is based on requiring that the admissible strategies satisfy a cone constraint. Unfortunately, in practice, constraints which can not be expressed as a cone constraint may also need to be imposed.

Finally, the method most closely related to the embedding method is based on using a Lagrange multiplier technique [11, 22]. Formally, this method requires that the problem can be posed as a convex optimization problem. This cannot be guaranteed in the case of the optimal execution problem discussed in [27], where the differential equations describing the underlying processes are nonlinear. It is interesting to observe that the final objective function in the Lagrange multiplier method has the same algebraic structure as the objective function in the embedding method.

1.3 Contributions of this paper

Although the theoretical results in [27] are important and practically useful, there is one additional complication which has not been addressed: it is computationally infeasible to compute the entire set $\mathcal{Y}_Q^\dagger$, since the embedding parameter $\gamma \in (-\infty, +\infty)$. In practice, we can only compute a solution of the embedded optimization problem for a set of finitely sampled embedding parameter values. Assume that $\Gamma_k \subset (-\infty, +\infty)$ is the set of sampled γ values at the sampling discretization level k, and denote the MV finite set corresponding to Γ_k by $(\mathcal{Y}_Q^\dagger)^k$. We assume that the index k is positively proportional to the number of finite values of γ used in computation. A conjecture made in [27] is that any reasonable finite sampling method for γ, such as systematically refining uniform grids, results in the set $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$ converging to, possibly a subset of, the set $\mathcal{S}(\mathcal{Y}_Q^\dagger)$ (or equivalently, the set $\mathcal{Y}_P$). However, this is by no means obvious, due to the fact that we use supporting hyperplanes of $(\mathcal{Y}_Q^\dagger)^k$ to determine $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$. Given the importance of the embedding technique and its popularity in multi-period MV optimization, it is highly desirable to mathematically establish the validity of this conjecture. In other words, it is necessary to analyze asymptotic properties of SOPs with respect to the discretization of the embedding parameter. As a result, we can develop a post-processing technique for the computed $(\mathcal{Y}_Q^\dagger)^k$ to produce $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$.

The main contributions of this paper can be summarized as follows.

- We prove that, under mild assumptions on sampling schemes, as $k \to +\infty$, every limit point in $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$ is a point in $\mathcal{S}(\mathcal{Y}_Q^\dagger)$. That is, every point in $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$ obtained from numerically solving the embedding problem with sufficiently large k can be an accurate approximation to an MV scalarization optimal point.

- The above result and the results developed in [27] form a numerical framework for determining valid (i.e. not spurious) points on the original efficient frontier. As such, these results complement the theoretical results of the embedding technique developed in [21, 33] for multi-period or continuous time MV optimization. Note that we do not require convexity of the original problem.

- We illustrate the theoretical findings of this paper for an MV asset-liability problem under jump diffusions. In this case, the frontier generated by the embedding technique *does* contain spurious points. This example highlights the importance of our post-processing numerical method.

To focus on the main issue of embedding parameter discretization, we assume that each point in $\mathcal{Y}_Q^\dagger$ is the exact solution of an embedded optimization problem corresponding to an embedding parameter.

The remainder of this paper is organized as follows. In Section 2, we summarize relevant major findings in [27] for removing spurious points which are used in subsequent sections of the paper. The main theoretical results on asymptotic convergence of the computed MV embedded post-processed set $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$ are presented in Section 3. In Section 4, we provide a numerical example for the MV asset-liability management under jump diffusions. This requires solution of an HJB partial integro-differential equation (PIDE). In Section 5 we discuss application of our main results to other techniques (such as a Monte Carlo, Backward Stochastic Differential Equation formulation) for numerically solving the embedded control problem. Section 6 concludes the paper and outlines possible future work.

2 Removal of spurious points by identifying SOPs

We first briefly summarize the main results in [27], following notation used in [27]. We denote by $X(t)$ the underlying multi-dimensional stochastic process and by x a state of the stochastic system. We use $c(\cdot)$ to denote the control, representing a strategy, as a function of the current state, i.e. $c(\cdot) : (X(t),t) \mapsto c = c(X(t),t)$. Furthermore, we denote by W_t the total wealth at time t. Let $E_{c(\cdot)}^{x,t}[W_T]$ and $Var_{c(\cdot)}^{x,t}[W_T]$ respectively denote the expectation and the variance of the terminal wealth W_T conditional on the initial state (x,t) and on the control $c(\cdot)$.

2.1 MV Pareto optimal set

Since we are mainly interested in identifying spurious points on an efficient frontier, we analyze MV optimality in terms of time T achievable expected value and variance of the wealth. We first introduce a few definitions.

Definition 2.1. *Let $(x_0,0) = (X(t=0), t=0)$ denote the initial state. Let*

$$\mathcal{Y} = \left\{ (Var_{c(\cdot)}^{x_0,0}[W_T], E_{c(\cdot)}^{x_0,0}[W_T]) : c(\cdot) \text{ admissible } \right\} \tag{2.1}$$

denote the **achievable MV objective set** *and $\overline{\mathcal{Y}}$ denote its closure.*

Definition 2.2. *A point $(\mathcal{V}_*, \mathcal{E}_*) \in \overline{\mathcal{Y}}$ is a* **Pareto (optimal) point** *if there exists no admissible strategy $c(\cdot)$ such that*

$$\begin{array}{l} E_{c(\cdot)}^{x_0,0}[W_T] \geq \mathcal{E}_* \\ Var_{c(\cdot)}^{x_0,0}[W_T] \leq \mathcal{V}_* \ , \end{array}$$

and at least one of the inequalities in equation (2.2) is strict. We denote by $\mathcal{P}$ the set of Pareto (optimal) points. Note that $\mathcal{P} \subseteq \overline{\mathcal{Y}}$.

Although the above definitions are intuitive, determining the points in $\mathcal{P}$ requires solving a difficult multi-objective optimization problem, which includes two conflicting criteria. A standard scalarization method can be used to combine the two criteria into an optimization problem with a single objective. More specifically, for an arbitrary scaler $\mu > 0$, we first define $\mathcal{Y}_{P(\mu)}$ to be the set of scalarization optimal points for the parameter μ,

$$\mathcal{Y}_{P(\mu)} \quad = \quad \{(\mathcal{V}_*, \mathcal{E}_*) \in \overline{\mathcal{Y}} \colon \mu\mathcal{V}_* - \mathcal{E}_* = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} (\mu\mathcal{V} - \mathcal{E})\} \ . \tag{2.2}$$

We then define the **MV scalarization optimal set**, denoted by $\mathcal{Y}_P$, as

$$\mathcal{Y}_P = \bigcup_{\mu>0} \mathcal{Y}_{P(\mu)}. \tag{2.3}$$

where we note that it is possible for $\mathcal{Y}_{P(\mu)}$ to be empty for some $\mu > 0$.

We recognize the difference between the set of all MV Pareto optimal points $\mathcal{P}$ and the set of MV scalarization optimal points $\mathcal{Y}_P$ defined in equation (2.3). In general, $\mathcal{Y}_P \subseteq \mathcal{P}$. However, the converse may not hold, if the achievable MV objective set $\mathcal{Y}$ is not convex. As in [27], we restrict our attention to determining $\mathcal{Y}_P$.

2.2 Embedding methods

As noted in [21, 33], the presence of the variance term in equation (2.2) causes difficulty, if we attempt to determine $\mathcal{Y}_{P(\mu)}$ by directly solving for the associated value function using dynamic programming. To overcome this difficulty, we can make use of the main result in [21, 33] concerning the embedding technique. Similar to $\mathcal{Y}_P$, we can describe embedding optimality in terms of an achievable objective point $(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}$.

Definition 2.3 (Embedded MV objective set)**.** *The **embedded MV objective set** $\mathcal{Y}_Q$ is defined by*

$$\mathcal{Y}_Q = \bigcup_{-\infty<\gamma<+\infty} \mathcal{Y}_{Q(\gamma)}. \tag{2.4}$$

where

$$\mathcal{Y}_{Q(\gamma)} = \big\{(\mathcal{V}_*, \mathcal{E}_*) \in \overline{\mathcal{Y}} : \mathcal{V}_* + \mathcal{E}_*^2 - \gamma\mathcal{E}_* = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} \mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E}\big\}. \tag{2.5}$$

Remark 2.1 (Nonemptyness of $\mathcal{Y}_{Q(\gamma)}$)**.** *Write $\mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E}$ as $\mathcal{V} + (\mathcal{E} - \gamma/2)^2 - \gamma^2/4$. Noting that variance $\mathcal{V} \geq 0$, we have that $\mathcal{V}+\mathcal{E}^2-\gamma\mathcal{E}$ is bounded from below for any γ. If $\mathcal{Y} \neq \emptyset$, then $\mathcal{Y}_{Q(\gamma)} \neq \emptyset$ $\forall\gamma$. Thus $\inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} \mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E}$ exists and the closure $\overline{\mathcal{Y}}$ contains $(\mathcal{V}_*, \mathcal{E}_*)$.*

Remark 2.2 (Dynamic programming form)**.** *Since*

$$\begin{aligned} \mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E} &= E^{x_0,0}_{c(\cdot)}[W_T^2] - (E^{x_0,0}_{c(\cdot)}[W_T])^2 + (E^{x_0,0}_{c(\cdot)}[W_T])^2 - \gamma E^{x_0,0}_{c(\cdot)}[W_T] &(2.6)\\ &= E^{x_0,0}_{c(\cdot)}[W_T^2 - \gamma W_T] &(2.7)\end{aligned}$$

then we can write equation (2.5) in standard control form

$$\inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} \mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E} = \inf_{c(\cdot)} E^{x_0,0}_{c(\cdot)}[W_T^2 - \gamma W_T] \tag{2.8}$$

which is now amenable to solution by a dynamic programming technique.

Definition 2.4. *A point $(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}_Q$ is a spurious point if $(\mathcal{V}, \mathcal{E}) \notin \mathcal{Y}_P$.*

We also introduce the concept of scalarization optimal points (SOPs) with respect to a set.

Definition 2.5. *Let $\mathcal{X}$ be a non-empty subset of $\overline{\mathcal{Y}}$. We define*

$$\mathcal{S}_\mu(\mathcal{X}) = \big\{(\mathcal{V}_*, \mathcal{E}_*) \in \overline{\mathcal{X}} : \mu\mathcal{V}_* - \mathcal{E}_* = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{X}} \mu\mathcal{V} - \mathcal{E}\big\}, \tag{2.9}$$

where $\overline{\mathcal{X}}$ is the closure of $\mathcal{X}$. We call a point in $\mathcal{S}_\mu(\mathcal{X})$ a scalarization optimal point (SOP) w.r.t. $(\mathcal{X}, \mu)$. We also define

$$\mathcal{S}(\mathcal{X}) = \big\{(\mathcal{V}_*, \mathcal{E}_*) : (\mathcal{V}_*, \mathcal{E}_*) \text{ is an SOP w.r.t. } (\mathcal{X}, \mu) \text{ for some } \mu > 0\big\}. \tag{2.10}$$

We refer to $(\mathcal{V}_0, \mathcal{E}_0) \in \mathcal{S}(\mathcal{X})$ as **SOP w.r.t.** *$\mathcal{X}$.*

Geometrically speaking, an SOP with respect to a set is a point at which there exists a supporting hyperplane with a positive slope for that set. We make the following assumption on the achievable objective set $\mathcal{Y}$.

Assumption 2.1 (Nonemptyness). *We assume that $\mathcal{Y}$ is a non-empty subset of $\{(\mathcal{V},\mathcal{E}) \in \mathbf{R}^2 : \mathcal{V} \geq 0\}$ and that there exists a positive scalarization parameter $\mu_E > 0$ such that $\mathcal{S}_{\mu_E}(\mathcal{Y}) \neq \emptyset$.*

Lemma 2.1 (Nonemptyness of $\mathcal{S}_\mu(\mathcal{Y}), \mu \geq \mu_E$). *If Assumption 2.1 holds, then $\forall \mu \geq \mu_E$, $\mathcal{S}_\mu(\mathcal{Y}) \neq \emptyset$.*

Proof. Let $(\mathcal{V}_E,\mathcal{E}_E) \in \mathcal{S}_{\mu_E}(\mathcal{Y})$. For any given $\mu \geq \mu_E$, consider $\forall(\mathcal{V},\mathcal{E}) \in \mathcal{Y}$,

$$\mu\mathcal{V} - \mathcal{E} \geq \mu_E\mathcal{V} - \mathcal{E} \geq \mu_E\mathcal{V}_E - \mathcal{E}_E. \tag{2.11}$$

Hence, $\mathcal{S}_\mu(\mathcal{Y}) \neq \emptyset$ for all $\mu \geq \mu_E$. □

Remark 2.3. *Note that $\mathcal{V} \geq 0$ always holds since the variance is non-negative. In [27], to ensure that $\mathcal{Y}_{P(\mu)} \neq \emptyset$, a stronger assumption was made that $\forall(\mathcal{V},\mathcal{E}) \in \mathcal{Y}, \mathcal{E} \leq C_E$, where C_E is a constant. Due to Lemma 2.1, the results in [27] hold under Assumption 2.1.*

The embedding result of [21, 33] is summarized in Theorem 2.1, under the weaker Assumption 2.1. An important implication of Theorem 2.1 is that $\mathcal{Y}_P \subseteq \mathcal{Y}_Q$.

Theorem 2.1 (Embedding result – Theorem 4.4 in [27]). *If Assumption 2.1 holds and $\mu \geq \mu_E$, then $\mathcal{S}_\mu(\mathcal{Y}) \neq \emptyset$. Assume $(\mathcal{V}_0,\mathcal{E}_0) \in \mathcal{Y}_{P(\mu)}$. Then*

$$\mu\mathcal{V}_0 - \mathcal{E}_0 = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} \mu\mathcal{V} - \mathcal{E} \tag{2.12}$$

and

$$\mathcal{V}_0 + \mathcal{E}_0^2 - \gamma\mathcal{E}_0 = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} \mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E}, \quad \textit{i.e.} \;\; (\mathcal{V}_0,\mathcal{E}_0) \in \mathcal{Y}_{Q(\gamma)}, \tag{2.13}$$

where

$$\gamma = \frac{1}{\mu} + 2\mathcal{E}_0. \tag{2.14}$$

For subsequent analysis, we present the following uniqueness property of the embedded MV objective set $\mathcal{Y}_{Q(\gamma)}$ established in [27] .

Theorem 2.2 (Uniqueness of $\mathcal{Y}_{Q(\gamma)}$ – Theorem 4.8 in [27]). *If $(\mathcal{V},\mathcal{E}) \in \mathcal{S}(\mathcal{Y}_Q)$, then there exists γ such that $(\mathcal{V},\mathcal{E}) \in \mathcal{Y}_{Q(\gamma)}$ and $\mathcal{Y}_{Q(\gamma)}$ is a singleton.*

The following result from [27] indicates that spurious points can be identified as not being SOP with respect to the embedded MV objective set.

Theorem 2.3 (Theorem 4.7 in [27]). *The SOPs w.r.t. $\mathcal{Y}_Q$ are the same as the SOPs w.r.t. $\mathcal{Y}$, i.e.*

$$\mathcal{S}(\mathcal{Y}_Q) = \mathcal{Y}_P = \mathcal{S}(\mathcal{Y}). \tag{2.15}$$

Theorem 2.3 demonstrates that it is possible to generate the original MV SOP set $\mathcal{Y}_P$ from the embedded MV objective set $\mathcal{Y}_Q$. More specifically, a spurious point in $\mathcal{Y}_Q$ is a point at which there does not exist a supporting hyperplane with positive slope for $\mathcal{Y}_Q$. Excluding all these spurious points from $\mathcal{Y}_Q$, we obtain $\mathcal{S}(\mathcal{Y}_Q)$, and hence, $\mathcal{Y}_P$.

Remark 2.4 (Existence of Spurious Points). *Spurious points can arise from two distinct causes. If the original MV problem is not convex, then it is easily seen that spurious points can be generated. However, even if the MV problem is convex, a numerical algorithm based on minimizing the objective function (2.5) may produce spurious points (as defined in Definition 2.4). This is because numerically computed points in* $\mathcal{Y}_Q$ *may not satisfy all the conditions (2.12)–(2.14). If we have a closed form solution, as in [21, 33], then necessary condition (2.14) (where* μ *satisfies (2.12)) can be explicitly imposed, so that this situation does not arise. However, given an arbitrary point in* $\mathcal{Y}_Q$*, generated by a numerical algorithm, then we cannot verify that condition (2.14) is satisfied, without examining the entire set* $\mathcal{Y}_Q$*. However, we can ensure that both types of spurious points can be eliminated if we consider only the S.O.Ps w.r.t* $\mathcal{Y}_Q$ *as in Theorem 2.3. Note that spurious points were not generated in[21, 33], since the MV problems considered were convex, and the closed-form solutions enabled imposition of condition (2.14).*

2.3 Removal of spurious points with respect to the computed embedded MV set

However, Theorem 2.3 cannot be directly used in a numerical algorithm for construction of $\mathcal{S}(\mathcal{Y}_Q)$, since the entire set $\mathcal{Y}_Q$ is not available in practice. There are two aspects of incompleteness. The first is the incompleteness due to availability of only a single solution for each γ. For each embedding parameter γ, $-\infty < \gamma < +\infty$, a numerical algorithm applied to the embedded problem can generate only a single embedded MV point $(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}_{Q(\gamma)}$. In this case, it is not obvious that the single embedded MV point generated by our algorithm will satisfy all the conditions (2.12)–(2.14). This first aspect of incompleteness is addressed in [27]; the relevant result is summarized below. The second aspect of incompleteness is due to the fact that, in practice, only a finite number of γ values can be used to approximate the set $\mathcal{Y}_Q$. This aspect of incompleteness is the focus of this paper, and is discussed in Section 3. We define the **computed MV embedded objective set**, denoted by $\mathcal{Y}_Q^\dagger$, as follows.

Definition 2.6 (Computed MV embedded objective set). *Let* $\mathcal{Y}^\dagger_{Q(\gamma)}$ *be a singleton subset of* $\mathcal{Y}_{Q(\gamma)}$*. Specifically* $\mathcal{Y}^\dagger_{Q(\gamma)}$ *contains either*

- *the unique single point which is SOP w.r.t.* $\mathcal{Y}_Q$ *if* $\mathcal{Y}_{Q(\gamma)}$ *is the singleton set containing a point SOP w.r.t.* $\mathcal{Y}_Q$*, or*
- *an arbitrarily selected single point of* $\mathcal{Y}_{Q(\gamma)}$ *otherwise.*

The computed MV objective set is then defined as

$$\mathcal{Y}_Q^\dagger = \bigcup_{-\infty<\gamma<+\infty} \mathcal{Y}^\dagger_{Q(\gamma)}.$$

The following theorem shows that $\mathcal{Y}_P$ can be generated from $\mathcal{Y}_Q^\dagger$.

Theorem 2.4 (Theorem 5.4 in [27]). *Suppose Assumption 2.1 holds. Then*

$$\mathcal{S}(\mathcal{Y}_Q^\dagger) = \mathcal{Y}_P = \mathcal{S}(\mathcal{Y}) \ . \tag{2.16}$$

Following immediately from Theorem 2.4, Lemma 2.1, we have Corollary 2.1.

Corollary 2.1. *Suppose Assumption 2.1 holds. Then* $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset, \forall \mu \geq \mu_E$.

An important implication of Theorem 2.4 is that, given an MV point $(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}_Q^\dagger$, we can determine whether it is in $\mathcal{Y}_P$ by checking whether it is an SOP with respect to $\mathcal{Y}_Q^\dagger$.

3 Asymptotic properties of sets of SOPs with respect to the embedding parameter sampling

It is important to note that the procedure given in [27], described so far, requires the entire set $\mathcal{Y}_Q^\dagger$ to be available, i.e. an embedded MV point for each $\gamma \in (-\infty, +\infty)$. However, we typically solve the embedded problem for each fixed γ by numerically solving the associated HJB equation. Hence, in practice, we can only approximate $\mathcal{Y}_Q^\dagger$ for a finite number of γ values. More specifically, we approximate $\mathcal{Y}_Q^\dagger$ using a finite set of γ values, each of which yields a solution to the embedded problem (2.13). As a result, a sampling discretization for γ needs to be implemented. In addition, to assess convergence of the approximation of $\mathcal{Y}_Q^\dagger$, a sequence of samplings of γ needs to be computed. To capture this, we denote by Γ_k the finite discrete set of sampled γ values at the sampling discretization level k. Examples of methods for constructing Γ_k are given in §3.2. Let

$$(\mathcal{Y}_Q^\dagger)^k = \bigcup_{\gamma \in \Gamma_k} \mathcal{Y}_{Q(\gamma)}^\dagger \tag{3.1}$$

denote the set of all computed MV embedded points using the sampling set Γ^k. Note that

$$(\mathcal{Y}_Q^\dagger)^k \subseteq \mathcal{Y}_Q^\dagger. \tag{3.2}$$

In addition, we need to construct the SOPs of $(\mathcal{Y}_Q^\dagger)^k$, i.e. $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$. A simple method which constructs $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$ is described in Algorithm 3.1. Theoretical justification of Algorithm 3.1 is

Algorithm 3.1 Post-processing algorithm to construct $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$ from $(\mathcal{Y}_Q^\dagger)^k$.

1: determine the set $(\mathcal{C})^k$ consisting of all the vertices of the convex hull of $(\mathcal{Y}_Q^\dagger)^k$;
2: determine the set $(\mathcal{U})^k$ consisting of upper-left boundary points of $(\mathcal{C})^k$;
3: return $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k) \equiv (\mathcal{U})^k$.

given in [27]. In [27], it is conjectured that for sufficiently large k, the points in $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$ sufficiently well approximate the points in $\mathcal{S}((\mathcal{Y}_Q^\dagger)) = \mathcal{Y}_P$. In this section, we analyze convergence properties of $(\mathcal{Y}_Q^\dagger)^k$ as $k \to +\infty$. Our aim is to show that, as $k \to +\infty$, every limit point of a sequence $\{(\mathcal{E}_k, \mathcal{V}_k)\}$, $(\mathcal{E}_k, \mathcal{V}_k) \in \mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)$, is a point in $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$.

Remark 3.1 (Intuitive explanation of Algorithm 3.1)**.** *Note that* $(\mathcal{Y}_Q^\dagger)^k$ *is a finite set of points. If a point is in* $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$*, then there exists a supporting hyperplane with positive slope at that point. These points are also the vertices of the upper left convex hull of* $(\mathcal{Y}_Q^\dagger)^k$ *[27]. The vertices of the upper left convex hull of* m *points can be computed in* $O(m \log m)$ *time, using, for example, the algorithm in [2].*

3.1 Preliminaries

In preparation for the convergence analysis, we first establish a few technical lemmas. Recall that

$$\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \quad = \quad \big\{(\mathcal{V}_*, \mathcal{E}_*) \in \overline{\mathcal{Y}_Q^\dagger} : \mu\mathcal{V}_* - \mathcal{E}_* = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}_Q^\dagger} \mu\mathcal{V} - \mathcal{E}\big\}. \tag{3.3}$$

One thing that makes the asymptotic analysis challenging is that, for a given μ, there can be multiple points in $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$. We handle this difficulty by examining the minimum element of $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ for each given μ.

Definition 3.1. *For $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$, $\mu > 0$, we define the minimum element as $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu))$ where*

$$(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger),\ \mathcal{V}^{\min}(\mu) \leq \mathcal{V},\ \mathcal{E}^{\min}(\mu) \leq \mathcal{E},\ \forall (\mathcal{V}, \mathcal{E}) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger). \tag{3.4}$$

3.1.1 Minimum element of $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$

Since any point in $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ lies on a supporting hyperplane with a slope $\mu > 0$, we immediately have the following Lemma concerning the existence and uniqueness of the minimum element of $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$.

Lemma 3.1. *Assume that $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ for $\mu > 0$. Then there exists an unique minimum $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu))$ for $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$. In addition,*

$$\mathcal{V}^{\min}(\mu) = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)} \mathcal{V}\ ,$$
$$\mathcal{E}^{\min}(\mu) = \mu\mathcal{V}^{\min}(\mu) - \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}_Q^\dagger} \mu\mathcal{V} - \mathcal{E}\ .$$

Proof. Let $\mu > 0$ be given. Since $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$, there exists the unique value

$$f_0(\mu) = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}_Q^\dagger} \mu\mathcal{V} - \mathcal{E}. \tag{3.5}$$

Then,

$$\mu\mathcal{V} - \mathcal{E} = f_0(\mu), \quad \forall (\mathcal{V}, \mathcal{E}) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger). \tag{3.6}$$

To show existence and uniqueness of $\mathcal{V}^{\min}(\mu)$, we note that $\mathcal{V} \geq 0$, i.e. $\mathcal{V}$ is bounded below. Hence, $\inf_{(\mathcal{V},\mathcal{E})\in\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)} \mathcal{V}$ exists. Let

$$\mathcal{V}^{\min}(\mu) = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)} \mathcal{V}. \tag{3.7}$$

Clearly, $\mathcal{V}^{\min}(\mu)$ is unique. In addition, we have

$$\mathcal{V}^{\min}(\mu) \leq \mathcal{V}, \ \ \forall\ (\mathcal{V}, \mathcal{E}) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger).$$

Now, we show existence and uniqueness of quantity $\mathcal{E}^{\min}(\mu)$. To this end, note that, there exists a sequence $\{(\mathcal{V}_k,\mathcal{E}_k)\}$, $(\mathcal{V}_k,\mathcal{E}_k)\in\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$, such that

$$\lim_{k\to+\infty}\mathcal{V}_k=\mathcal{V}^{\min}(\mu).$$

For any sequence $\{(\mathcal{V}_k,\mathcal{E}_k)\}$, $(\mathcal{V}_k,\mathcal{E}_k)\in\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$, with $\lim_{k\to+\infty}\mathcal{V}_k=\mathcal{V}^{\min}(\mu)$, we have

$$\mu\mathcal{V}_k-\mathcal{E}_k=f_0(\mu).$$

Thus,

$$\lim_{k\to+\infty}\mathcal{E}_k=\lim_{k\to+\infty}(\mu\mathcal{V}_k)\ \ -f_0(\mu)=\mu\mathcal{V}^{\min}(\mu)-f_0(\mu).$$

Define the unique value

$$\mathcal{E}^{\min}(\mu)=\lim_{k\to+\infty}\mathcal{E}_k=\mu\mathcal{V}^{\min}(\mu)-f_0(\mu), \tag{3.8}$$

or equivalently,

$$\mu\mathcal{V}^{\min}(\mu)-\mathcal{E}^{\min}(\mu)=f_0(\mu). \tag{3.9}$$

By (3.3), (3.5) and (3.9), it follows that $(\mathcal{V}^{\min}(\mu),\mathcal{E}^{\min}(\mu))\in\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$. From (3.6) and (3.9), we have

$$\mathcal{E}^{\min}(\mu)=\mu\mathcal{V}^{\min}(\mu)-(\mu\mathcal{V}-\mathcal{E})\quad,\quad\forall\ (\mathcal{V},\mathcal{E})\in\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger).$$

Hence, Lemma 3.1 holds. □

3.1.2 Continuity of $(\mathcal{V}^{\min}(\mu),\mathcal{E}^{\min}(\mu))$

Now we show that $(\mathcal{V}^{\min}(\mu),\mathcal{E}^{\min}(\mu))$ is right-continuous in μ. In the following supporting Lemma, we first establish the monotonicity of $(\mathcal{V}^{\min}(\mu),\mathcal{E}^{\min}(\mu))$.

Lemma 3.2. *Assume $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)\neq\emptyset$. Let $(\mathcal{V}^{\min}(\mu),\mathcal{E}^{\min}(\mu))\in\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$, and $(\mathcal{V}^{\min}(\mu'),\mathcal{E}^{\min}(\mu'))\in\mathcal{S}_{\mu'}(\mathcal{Y}_Q^\dagger)$. If $\mu'>\mu$, then*

$$\mathcal{V}^{\min}(\mu')\le\mathcal{V}^{\min}(\mu)\quad\text{ and }\quad\mathcal{E}^{\min}(\mu')\le\mathcal{E}^{\min}(\mu). \tag{3.10}$$

Proof. From Corollary 2.1, $\mathcal{S}_{\mu'}(\mathcal{Y}_Q^\dagger)\neq\emptyset$. Since $(\mathcal{V}^{\min}(\mu'),\mathcal{E}^{\min}(\mu'))\in\mathcal{S}_{\mu'}(\mathcal{Y}_Q^\dagger)\subseteq\overline{\mathcal{Y}_Q^\dagger}$, there exists a sequence $\{(\mathcal{V}_k,\mathcal{E}_k)\}$, $(\mathcal{V}_k,\mathcal{E}_k)\in\mathcal{Y}_Q^\dagger$, such that

$$\lim_{k\to\infty}\mathcal{V}_k=\mathcal{V}^{\min}(\mu')\quad\text{and}\quad\lim_{k\to\infty}\mathcal{E}_k=\mathcal{E}^{\min}(\mu'). \tag{3.11}$$

Since $(\mathcal{V}^{\min}(\mu),\mathcal{E}^{\min}(\mu))\in\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$,

$$\mu\mathcal{V}^{\min}(\mu)-\mathcal{E}^{\min}(\mu)\le\mu\mathcal{V}-\mathcal{E},\quad\forall\ (\mathcal{V},\mathcal{E})\in\mathcal{Y}_Q^\dagger. \tag{3.12}$$

From (3.11) and (3.12), we have

$$\mu\mathcal{V}^{\min}(\mu)-\mathcal{E}^{\min}(\mu)\le\mu\mathcal{V}^{\min}(\mu')-\mathcal{E}^{\min}(\mu'). \tag{3.13}$$

Interchanging the role of μ and μ' in (3.13), we have

$$-(\mu'\mathcal{V}^{\min}(\mu) - \mathcal{E}^{\min}(\mu)) \leq -(\mu'\mathcal{V}^{\min}(\mu') - \mathcal{E}^{\min}(\mu')). \tag{3.14}$$

Adding (3.13) and (3.14) gives

$$(\mu - \mu')\mathcal{V}^{\min}(\mu) \leq (\mu - \mu')\mathcal{V}^{\min}(\mu') \Rightarrow (\mu - \mu')(\mathcal{V}^{\min}(\mu) - \mathcal{V}^{\min}(\mu')) \leq 0. \tag{3.15}$$

Since $\mu' > \mu$, it follows from (3.15) that $\mathcal{V}^{\min}(\mu') \leq \mathcal{V}^{\min}(\mu)$.

By first multiplying (3.13) and (3.14) with μ' and μ, respectively, then adding the resulting inequalities, we obtain

$$(\mu - \mu')(\mathcal{E}^{\min}(\mu) - \mathcal{E}^{\min}(\mu')) \leq 0.$$

It follows that $\mathcal{E}^{\min}(\mu') \leq \mathcal{E}^{\min}(\mu)$. □

Next we establish that $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ is right-continuous in μ.

Lemma 3.3. *Assume that $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ at $\mu = \mu_0$. Then, $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ are right-continuous in $[\mu_0, +\infty)$.*

Proof. From Corollary 2.1, $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ for all $\mu \geq \mu_0$. Let $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$, and $(\mathcal{V}^{\min}(\mu_0), \mathcal{E}^{\min}(\mu_0)) \in \mathcal{S}_{\mu_0}(\mathcal{Y}_Q^\dagger)$. Following Lemma 3.2, we have

$$\mathcal{E}^{\min}(\mu) \leq \mathcal{E}^{\min}(\mu_0), \quad \text{and} \quad \mathcal{V}^{\min}(\mu) \leq \mathcal{V}^{\min}(\mu_0), \quad \forall \mu \geq \mu_0. \tag{3.16}$$

Due to this monotonicity, there exists $(\mathcal{E}_L, \mathcal{V}_L) \in \overline{\mathcal{Y}_Q^\dagger}$ such that

$$\lim_{\mu \to \mu_0^+} \mathcal{E}^{\min}(\mu) = \mathcal{E}_L, \quad \text{and} \quad \lim_{\mu \to \mu_0^+} \mathcal{V}^{\min}(\mu) = \mathcal{V}_L. \tag{3.17}$$

To show right-continuity, we now establish that $\mathcal{E}_L = \mathcal{E}^{\min}(\mu)$ and $\mathcal{V}_L = \mathcal{V}^{\min}(\mu)$. From (3.16)-(3.17), we conclude that

$$\mathcal{E}_L \leq \mathcal{E}^{\min}(\mu_0); \quad \mathcal{V}_L \leq \mathcal{V}^{\min}(\mu_0). \tag{3.18}$$

Since

$$\mu\mathcal{V}^{\min}(\mu) - \mathcal{E}^{\min}(\mu) \leq \ \mu\mathcal{V} - \mathcal{E}, \quad \forall (\mathcal{V}, \mathcal{E}) \in \mathcal{Y}_Q^\dagger,$$

by letting $\mu \to \mu_0^+$ and using (3.17), we obtain

$$\mu_0\mathcal{V}_L - \mathcal{E}_L \leq \ \mu_0\mathcal{V} - \mathcal{E}, \quad \forall (\mathcal{V}, \mathcal{E}) \in \mathcal{Y}_Q^\dagger. \tag{3.19}$$

Since $(\mathcal{E}_L, \mathcal{V}_L) \in \overline{\mathcal{Y}_Q^\dagger}$, and $(\mathcal{V}^{\min}(\mu_0), \mathcal{E}^{\min}(\mu_0)) \in \mathcal{S}_{\mu_0}(\mathcal{Y}_Q^\dagger) \subseteq \mathcal{Y}_Q^\dagger$, it follows from (3.19) that

$$\mu_0\mathcal{V}_L - \mathcal{E}_L = \mu_0\mathcal{V}^{\min}(\mu_0) - \mathcal{E}^{\min}(\mu_0).$$

Hence, $(\mathcal{E}_L, \mathcal{V}_L) \in \mathcal{S}_{\mu_0}(\mathcal{Y}_Q^\dagger)$. By Lemma 3.1, we have

$$\mathcal{V}^{\min}(\mu_0) \leq \mathcal{V}_L, \quad \mathcal{E}^{\min}(\mu_0) \leq \mathcal{E}_L. \tag{3.20}$$

From (3.18) and (3.20), it follows that $\mathcal{E}_L = \mathcal{E}^{\min}(\mu)$ and $\mathcal{V}_L = \mathcal{V}^{\min}(\mu)$. □

3.1.3 Continuity and monotonicity of $\gamma^{\min}(\mu)$

We note that the embedding parameter $\gamma^{\min}(\mu)$, corresponding to the minimum element, also plays an important role in the asymptotic analysis. Next, we show that, assuming $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu))$ exists, the corresponding embedding parameter $\gamma^{\min}(\mu)$ is right-continuous and strictly decreasing in μ. First, we establish a supporting Lemma that relates the embedding parameter $\gamma^{\min}$ and the scalarization parameter μ.

Lemma 3.4. *Assume that* $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ *and that* $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$. *Then, there exists a unique* $\gamma^{\min}(\mu)$ *such that*

$$\begin{aligned} \gamma^{\min}(\mu) &= \frac{1}{\mu} + 2\mathcal{E}^{\min}(\mu), \\ &\quad \text{where } (\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{Y}^\dagger_{Q(\gamma^{\min}(\mu))}, \end{aligned} \tag{3.21}$$

and $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu))$ *is the unique point in* $\mathcal{Y}^\dagger_{Q(\gamma^{\min}(\mu))} \subseteq \mathcal{S}(\mathcal{Y}_Q^\dagger)$.

Proof. Since $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \subseteq \mathcal{S}(\mathcal{Y}_Q^\dagger)$, following Theorem 2.4, we have

$$(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{Y}_P = \mathcal{S}(\mathcal{Y}).$$

By Theorem 2.3, we have $\mathcal{S}(\mathcal{Y}) = \mathcal{S}(\mathcal{Y}_Q)$. Hence

$$(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{S}(\mathcal{Y}_Q).$$

Using Theorem 2.2, we have

$$(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu)) \in \mathcal{Y}_{Q(\gamma)} \quad \text{for some } \gamma,$$

and $\mathcal{Y}_{Q(\gamma)}$ is a singleton. Following Theorem 2.1, there exists an unique $\gamma^{\min}(\mu)$, which is defined below

$$\gamma^{\min}(\mu) = \frac{1}{\mu} + 2\mathcal{E}^{\min}(\mu).$$

By Definition 2.6, $\mathcal{Y}^\dagger_{Q(\gamma)}$ contains a single point, so that $(\mathcal{V}^{\min}(\mu), \mathcal{E}^{\min}(\mu))$ is the unique point in $\mathcal{Y}^\dagger_{Q(\gamma^{\min}(\mu))}$. This completes the proof. □

In the following Lemma, we establish the right-continuity and monotonicity of $\gamma^{\min}(\mu)$ in μ.

Lemma 3.5. *Assume that* $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ *for* $\mu = \mu_0 > 0$. *Then* $\gamma^{\min}(\mu)$ *is right-continuous and strictly decreasing in* $[\mu_0, +\infty)$.

Proof. From Corollary 2.1, $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ for any $\mu \in [\mu_0, +\infty)$, and from Lemma 3.3, $\mathcal{E}^{\min}(\mu)$ is right-continuous in $[\mu_0, +\infty)$. Thus, from (3.21), $\gamma^{\min}(\mu)$ is right-continuous in μ. To show monotonicity, note that, for any $\mu, \mu' \in (\mu_0, +\infty)$ and $\mu > \mu'$, we have (noting Lemma 3.2)

$$\frac{1}{\mu} < \frac{1}{\mu'} \quad ; \quad \mathcal{E}^{\min}(\mu) \leq \mathcal{E}^{\min}(\mu'). \tag{3.22}$$

Thus, from (3.21)-(3.22), we have that $\gamma^{\min}(\mu)$ is a strictly decreasing function of μ in $[\mu_0, +\infty)$. □

From Lemma 3.5, $\gamma^{\min}(\mu)$ is a strictly decreasing function of μ in $[\mu_0, +\infty)$. Following this, we immediately conclude that the inverse function ${\gamma^{\min}}^{-1}(\gamma)$, which yields an unique scalarization parameter, is left continuous at $\gamma^{\min}(\mu_0)$. Specifically, the inverse function ${\gamma^{\min}}^{-1}(\gamma)$ is uniquely defined in $(\underline{\gamma_0}^{\min}, \gamma^{\min}(\mu_0)]$ for some $\underline{\gamma_0}^{\min}$.

Next we analyze asymptotic properties of embedded MV points corresponding to a set Γ_k of sampled embedding parameter γ under some mild assumptions on Γ_k, see, e.g., Assumption 3.1.

Lemma 3.6. *Assume that $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ at $\mu = \mu_0$. Let $(\mathcal{V}^{\min}(\mu_0), \mathcal{E}^{\min}(\mu_0)) \in \mathcal{S}_{\mu_0}(\mathcal{Y}_Q^\dagger)$. Assume that there exists a monotonically increasing sequence of embedding parameters $\{\gamma_k\}$ satisfying*

$$\lim_{k\to\infty} \gamma_k = \gamma^{\min}(\mu_0) \quad ; \quad \gamma_k \geq \gamma_{k-1} \ .$$

Then for sufficiently large k, there exists a unique μ_k such that

$$\mu_k = {\gamma^{\min}}^{-1}(\gamma_k), \qquad \mu_k \geq \mu_0,$$

and

$$\lim_{k\to\infty} (\mathcal{V}^{\min}(\mu_k), \mathcal{E}^{\min}(\mu_k)) = (\mathcal{V}^{\min}(\mu_0), \mathcal{E}^{\min}(\mu_0)) \ .$$

Proof. Since $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ at $\mu = \mu_0$, from Lemma 3.5, $\gamma^{\min}(\mu)$ is monotonically decreasing and right-continuous in $[\mu_0, +\infty)$. Hence, from Lemma 3.5, for sufficiently large k, there exists a unique μ_k such that

$$\mu_k = {\gamma^{\min}}^{-1}(\gamma_k), \qquad \mu_k \geq \mu_0$$

and μ_k monotonically decreasing, such that

$$\lim_{k\to\infty} \mu_k = \mu_0$$

Note that $(\mathcal{V}^{\min}(\mu_0), \mathcal{E}^{\min}(\mu_0)) \in \mathcal{S}_{\mu_0}(\mathcal{Y}_Q^\dagger)$ is unique. In addition there exists unique

$$(\mathcal{V}^{\min}(\mu_k), \mathcal{E}^{\min}(\mu_k)) \in \mathcal{S}_{\mu_k}(\mathcal{Y}_Q^\dagger).$$

Furthermore,

$$\lim_{k\to\infty} (\mathcal{V}^{\min}(\mu_k), \mathcal{E}^{\min}(\mu_k)) = (\mathcal{V}^{\min}(\mu_0), \mathcal{E}^{\min}(\mu_0)),$$

which follows from the right-continuity of $(\mathcal{V}^{\min}(\mu_k), \mathcal{E}^{\min}(\mu_k))$ in $[\mu_0, +\infty)$. □

We conclude this subsection with a lemma which can be used to identify possible spurious points by examining only a subinterval of values of the embedding parameter γ.

Lemma 3.7. *Let Assumption 2.1 hold. Assume that there exists $\mathcal{E}^*$ such that*

$$\mathcal{E}^* = \liminf_{\mu\to+\infty} \{\mathcal{E}_\mu : (\mathcal{V}_\mu, \mathcal{E}_\mu) \in \mathcal{S}_\mu(\mathcal{Y})\} \ . \tag{3.23}$$

Then for any $\hat{\mu} > \mu_E$, there exists $\hat{\gamma}$, $-\infty < \hat{\gamma} < \infty$, such that

$$\hat{\gamma} = \frac{1}{\hat{\mu}} + 2\mathcal{E}^{\min}(\hat{\mu}). \tag{3.24}$$

where $(\mathcal{V}^{\min}(\hat{\mu}), \mathcal{E}^{\min}(\hat{\mu})) \in \mathcal{S}_{\hat{\mu}}(\mathcal{Y})$ and $(\mathcal{V}^{\min}(\hat{\mu}), \mathcal{E}^{\min}(\hat{\mu})) \in \mathcal{Y}_{Q(\hat{\gamma})}$. In addition

$$\hat{\gamma} > 2\mathcal{E}^*.$$

Proof. Since Assumption 2.1 holds, $\exists \mu_E > 0$ such that $\mathcal{S}_{\mu_E}(\mathcal{Y}) \neq \emptyset$. Hence $(\mathcal{V}^{\min}(\hat{\mu}), \mathcal{E}^{\min}(\hat{\mu})) \in \mathcal{S}_{\hat{\mu}}(\mathcal{Y})$ for $\hat{\mu} \geq \mu_E$. By Theorem 2.1, there exists $\hat{\gamma}$, $-\infty < \hat{\gamma} < \infty$, such that

$$\hat{\gamma} = \frac{1}{\hat{\mu}} + 2\mathcal{E}^{\min}(\hat{\mu}). \tag{3.25}$$

where $(\mathcal{V}^{\min}(\hat{\mu}), \mathcal{E}^{\min}(\hat{\mu})) \in \mathcal{Y}_{Q(\hat{\gamma})}$. From Lemma 3.2, $\mathcal{E}^{\min}(\hat{\mu})$ is non-increasing with respect to $\hat{\mu}$. Hence, from (3.23), we have that

$$\mathcal{E}^{\min}(\hat{\mu}) \geq \mathcal{E}^* \ .$$

Since $\hat{\mu} > 0$, from (3.24), we have $\hat{\gamma} > 2\mathcal{E}^{\min}(\hat{\mu}) \geq 2\mathcal{E}^*$. ☐

Remark 3.2. *Lemma 3.7 has the following important implication. Suppose that $(\mathcal{V}^{\min}, \mathcal{E}^{\min})$ is an optimal MV point for some embedding parameter $\hat{\gamma}$, i.e., $(\mathcal{V}^{\min}, \mathcal{E}^{\min}) \in \mathcal{Y}_{Q(\hat{\gamma})}$. If $\hat{\gamma} < 2\mathcal{E}^*$ and there exists no $\tilde{\gamma}$ such that $\tilde{\gamma} > 2\mathcal{E}^*$ and $(\mathcal{V}^{\min}, \mathcal{E}^{\min}) \in \mathcal{Y}_{Q(\tilde{\gamma})}$, then $(\mathcal{V}^{\min}, \mathcal{E}^{\min})$ is a spurious point.*

3.2 Asymptotic Convergence of $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$

Recall that Γ_k is the finite set of sampled γ values at the sampling discretization level k in the computation of the embedding technique. For subsequent analysis in the paper, we make the following assumption on Γ_k.

Assumption 3.1. *Assume that the sequence of finite set discretization refinements $\Gamma_k \subset (-\infty, \infty)$, $k = 1, 2, \ldots$, used in the computation of the embedding technique satisfies*

$$\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_k \subset \Gamma_{k+1} \subset \ldots . \tag{3.26}$$

In addition, for any fixed γ_*, there exists a monotonically increasing sequence $\{\gamma_{i_k}\}$, where $\gamma_{i_k} \in \Gamma_k$ and $\gamma_{i_k} \leq \gamma_*$, such that

$$\lim_{k \to \infty} \gamma_{i_k} \quad = \quad \gamma_* .$$

Remark 3.3. *It is straightforward to construct a sequence of discretization refinements Γ_k satisfying Assumption 3.1. As an example, we consider the following uniform discretization refinements when going from level k to level $k+1$.*

(1). Add new fine grid nodes between every two coarse grid nodes in Γ_k.

(2). Set $\max(\Gamma_{k+1}) = 2 * \max(|\max(\Gamma_k)|, |\min(\Gamma_k)|)$,

(3). Set $\min(\Gamma_{k+1}) = -2 * \max(|\max(\Gamma_k)|, |\min(\Gamma_k)|)$.

Next we investigate the asymptotic property of $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$. Recall that the set $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$ is the union of $\mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)$ for all positive μ where

$$\mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k) = \big\{(\mathcal{V}_*, \mathcal{E}_*) \in (\mathcal{Y}_Q^\dagger)^k : \mu\mathcal{V}_* - \mathcal{E}_* = \inf_{(\mathcal{V},\mathcal{E}) \in (\mathcal{Y}_Q^\dagger)^k} \mu\mathcal{V} - \mathcal{E}\big\}. \tag{3.27}$$

Lemma 3.8. *Assume* $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ *at* $\mu = \mu_0$. *Let* $(\mathcal{Y}_Q^\dagger)^k$ *be computed using the finite refinement* Γ_k, *where* Γ_k *satisfies Assumption 3.1. Then*

$$\lim_{k\to\infty}\left(\inf_{(\mathcal{V},\mathcal{E})\in(\mathcal{Y}_Q^\dagger)^k}(\mu_0\mathcal{V}-\mathcal{E})\right) \quad = \quad \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}_Q^\dagger}(\mu_0\mathcal{V}-\mathcal{E}). \tag{3.28}$$

Proof. Following Lemma 3.1, there exists $(\mathcal{V}^{\min}(\mu_0), \mathcal{E}^{\min}(\mu_0)) \in \mathcal{S}_{\mu_0}(\mathcal{Y}_Q^\dagger)$. Let

$$\begin{aligned} f &= \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}_Q^\dagger}(\mu_0\mathcal{V}-\mathcal{E}), \\ f_k &= \inf_{(\mathcal{V},\mathcal{E})\in(\mathcal{Y}_Q^\dagger)^k}(\mu_0\mathcal{V}-\mathcal{E}). \end{aligned} \tag{3.29}$$

From (3.2) and (3.26),

$$f_k \geq f, \quad \text{and} \quad f_{k+1} \leq f_k.$$

Hence, $\lim_{k\to\infty}\left(\inf_{(\mathcal{V},\mathcal{E})\in(\mathcal{Y}_Q^\dagger)^k}(\mu_0\mathcal{V}-\mathcal{E})\right) = \lim_{k\to\infty} f_k$ exists .

Next, we prove that (3.28) holds by contradiction. Suppose (3.28) does not hold. Since $f_k \geq f$, $\forall k$, and f_k is monotonically decreasing, it follows that $\exists\ \epsilon > 0$ s.t.

$$f \quad < f_k - \epsilon, \quad \forall k,$$

which implies that

$$f \leq \left(\inf_{(\mathcal{V},\mathcal{E})\in(\mathcal{Y}_Q^\dagger)^k}(\mu_0\mathcal{V}-\mathcal{E})\right) - \epsilon, \quad \forall k. \tag{3.30}$$

From Lemma 3.4, there exists $\gamma^{\min}(\mu_0)$ such that

$$\gamma^{\min}(\mu_0) = \frac{1}{\mu_0} + 2\mathcal{E}^{\min}(\mu_0).$$

From Assumption 3.1, there exists a monotonically increasing sequence $\{\gamma_k\}$ such that

$$\lim_{k\to\infty}\gamma_k \quad = \quad \gamma^{\min}(\mu_0), \quad \gamma_k \in \Gamma_k\ .$$

By Lemma 3.6, there exists a sequence $\{\mu_k\}$ and corresponding sequence $\{\gamma_k = \gamma^{\min}(\mu_k)\}$, $\gamma_k \in \Gamma_k$, such that

$$(\mathcal{V}^{\min}(\mu_k), \mathcal{E}^{\min}(\mu_k)) \in \mathcal{S}_{\mu_k}(\mathcal{Y}_Q^\dagger)\ ,$$

and

$$\lim_{k\to\infty}(\mathcal{V}^{\min}(\mu_k), \mathcal{E}^{\min}(\mu_k)) \quad = \quad (\mathcal{V}^{\min}(\mu_0), \mathcal{E}^{\min}(\mu_0)).$$

From Lemma 3.4, note that

$$(\mathcal{V}^{\min}(\mu_k), \mathcal{E}^{\min}(\mu_k)) \in \mathcal{S}_{\mu_k}(\mathcal{Y}_Q^\dagger) \quad = \quad \mathcal{Y}^\dagger_{Q(\gamma^{\min}(\mu_k))} \subset (\mathcal{Y}_Q^\dagger)^k\ . \tag{3.31}$$

Since $(\mathcal{V}^{\min}(\mu_0), \mathcal{E}^{\min}(\mu_0)) \in \mathcal{S}_{\mu_0}(\mathcal{Y}_Q^\dagger)$, we have

$$\mu_0 \mathcal{V}^{\min}(\mu_0) - \mathcal{E}^{\min}(\mu_0) = \inf_{(\mathcal{V},\mathcal{E}) \in \mathcal{Y}_Q^\dagger} \mu_0 \mathcal{V} - \mathcal{E} \ .$$

In other words, there exists a sequence of points (noting equation (3.31))

$$(\mathcal{V}_k, \mathcal{E}_k) = (\mathcal{V}^{\min}(\mu_k), \mathcal{E}^{\min}(\mu_k)) \quad \in \quad (\mathcal{Y}_Q^\dagger)^k$$

such that

$$\lim_{k \to \infty} \mu_0 \mathcal{V}_k - \mathcal{E}_k \quad = \quad \inf_{(\mathcal{V},\mathcal{E}) \in \mathcal{Y}_Q^\dagger} \mu_0 \mathcal{V} - \mathcal{E} = f,$$

which contradicts (3.30). $\square$

Now we establish an asymptotic property for $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$.

Theorem 3.1. *Let $(\mathcal{Y}_Q^\dagger)^k$ be computed using the finite refinement Γ_k of γ, where Γ_k satisfies Assumption 3.1. Assume that $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) \neq \emptyset$ for some $\mu > 0$. Let $(\mathcal{V}_k, \mathcal{E}_k) \in \mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)$. Let $(\mathcal{V}_*, \mathcal{E}_*)$ be a limit point of $\{(\mathcal{V}_k, \mathcal{E}_k)\}$. Then $(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$. If $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ is a singleton,i.e., $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger) = \{(\mathcal{V}_*, \mathcal{E}_*)\}$, then $\{(\mathcal{V}_k, \mathcal{E}_k)\}$ converges to $(\mathcal{V}_*, \mathcal{E}_*)$.*

Proof. Since $(\mathcal{Y}_Q^\dagger)^k \subseteq \mathcal{Y}_Q^\dagger$, $(\mathcal{V}_k, \mathcal{E}_k) \in \mathcal{Y}_Q^\dagger$. By Lemma 3.8, we have

$$\mu \mathcal{V}_* - \mathcal{E}_* \quad = \quad \lim_{k \to \infty} \mu \mathcal{V}_k - \mathcal{E}_k = \inf_{(\mathcal{V},\mathcal{E}) \in \mathcal{Y}_Q^\dagger} \mu \mathcal{V} - \mathcal{E}.$$

Hence, any limit point $(\mathcal{V}_*, \mathcal{E}_*)$ of $\{(\mathcal{V}_k, \mathcal{E}_k)\}$ is in $\mathcal{S}(\mathcal{Y}_Q^\dagger)$. If $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ is a singleton, it follows that $\{(\mathcal{V}_k, \mathcal{E}_k)\}$ converges to $\{(\mathcal{V}_*, \mathcal{E}_*)\} = \mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$. $\square$

Remark 3.4. *Theorem 3.1 implies that every limit point of a sequence in $\mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)$ converges to a point in $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ as the refinement level $k \to +\infty$. However, the converse is not true in general. More specifically, there may exist points in $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ which are not a limit point of $\{\mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)\}$. This can occur if there are three or more points in $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ for a fixed μ. In this case, only two of these points certainly are limit points of points in $\mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)$. In Figure 3.1, a pictorial illustration of this case is presented. In this illustration, for a fixed μ, there are five points in $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$, only two of which, namely the first and the fifth (from left to right), are certain to be limit points of points in $\mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)$.*

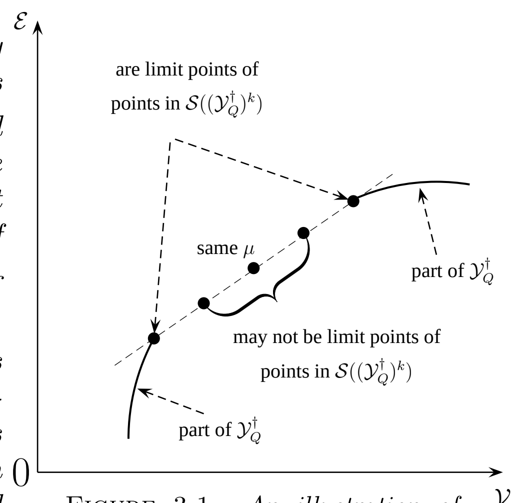


FIGURE 3.1: *An illustration of situations where there exist points in $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)$ which may not be limit points of points in $\mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)$.*

4 An MV optimal asset-liability example

In this section, we illustrate, using an MV optimal asset-liability example, the asymptotic relationship of the solution set $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$ corresponding to a sampling of embedding parameter Γ_k and the solution set $\mathcal{S}(\mathcal{Y}_Q^\dagger)$ corresponding to the full embedding parameter $(-\infty, +\infty)$.

We consider an MV asset allocation problem in which an investor dynamically adjusts positions in a risk-free asset, e.g. a bond, and a risky asset, e.g. a stock, to maximize the expected wealth of the investment portfolio, given a target level of risk. We refer the reader to [7, 21, 25, 28, 29, 33] and references therein, for a more detailed discussion on MV portfolio allocation.

It is common for investment institutions, such as pension funds or banks, to incorporate liabilities into portfolio allocation decisions. These asset-liability problems can be formulated as a multi-criteria MV optimization problem. This problem can then be solved via the embedding technique [31].

In our illustrating example, we focus on a typical case of an asset-liability problem under MV criteria where (a) the underlying risky asset follows a jump-diffusion, and (b) the liabilities are of the form of deterministic cash outflows. As a concrete example, we can consider the problem faced by a university endowment which is invested in risky assets, yet must fund fixed cash flows each year (e.g., an endowed chair).

More specifically, at each instant of a pre-determined set of dates, the investor (i) first withdraws an amount, subject to certain inflation rate, from the risk-free asset, and (ii) then rebalances the portfolio. We assume in the following that there is a leverage constraint and that trading must immediately cease if the investor is insolvent. We refer readers to [13] for discussions of constraints on (continuous time) MV portfolio allocation.

4.1 Underlying processes

We denote by S_t and B_t the *amounts* invested in the risky and the risk-free assets, respectively. For use later in the paper, define $t^- = t - \epsilon, \quad t^+ = t + \epsilon$, where $\epsilon \to 0^+$, i.e. t^- and t^+ respectively are instants of time just before and after the (forward) time t.

Under the objective measure, assume that S_t follows the process

$$\frac{dS_t}{S_{t^-}} = (\eta - \lambda\kappa)dt + \sigma dZ_t + d\left(\sum_{i=1}^{\pi_t}(\xi_i - 1)\right), \tag{4.1}$$

where dZ_t is the increment of a Wiener process, η is the real world drift rate, and σ is the volatility. In addition, π_t is a Poisson process with positive intensity parameter λ, and ξ_i are independent and identically distributed positive random variables having distribution (4.2). When a jump occurs, we have $S_{t^+} = \xi_i S_{t^-}$. As a specific example, consider ξ following a log-normal distribution $p(\xi)$ given by [24]

$$p(\xi) = \frac{1}{\sqrt{2\pi}\zeta\xi}\exp\left(-\frac{(\log(\xi) - \nu)^2}{2\zeta^2}\right), \tag{4.2}$$

with parameters ζ and ν. We have $E[\xi] = \exp(\nu + \zeta^2/2)$, where $E[\cdot]$ denotes the expectation operator, and $\kappa = E[\xi] - 1$.

We assume the dynamics of the risk-free asset B_t follows

$$dB_t = rB_t\, dt, \tag{4.3}$$

where r is the risk-free rate. We make the assumption that $\eta > r$, hence, it is never optimal (in an MV setting) to short stock. As a result, the amount invested in the risky asset is always nonnegative, i.e. $S_t \geq 0$. However, we allow short positions in the risk-free asset, i.e. it is possible that $B_t < 0$.

We consider the set of pre-determined times, referred to as *event times*,

$$t_1 < \ldots < t_M = T, \tag{4.4}$$

where T denotes the time horizon of the investment. We also denote by $t_0 = 0$ the inception time of the investment. We assume that there is no withdrawal at time t_0. At each event time t_i, $i = 1, \ldots, M$, the investor (i) first withdraws an amount of cash, denoted by a_i, from the risk-free asset, and (ii) then rebalances the portfolio. Here, the withdrawal amount a_i at the event time t_i is computed by $a_i = a(t_i - t_{i-1})e^{f \times t_i}$, where a is the (continuous) constant withdrawal rate, $t_i - t_{i-1}$ denotes the time interval between two event times t_i and t_{i-1}, and f is a (constant) inflation rate.

4.2 Liquidation value

In the remainder of the paper, let $X_t = (S_t, B_t)$ denote the multi-dimensional process and $x = (s, b)$ denote the state of the system. We denote by $W_t \equiv W(S_t, B_t) = S_t + B_t$, $t \leq T$, the total liquidation value at time t of the investor's portfolio. For use later in the paper, we define the solvency region, denoted by $\mathcal{N}$, as

$$\mathcal{N} \;=\; \{(s, b) \in [0, \infty) \times (-\infty, +\infty) : W(s, b) > 0\} \,. \tag{4.5}$$

The bankruptcy (insolvency) region, denoted by $\mathcal{B}$, is defined as

$$\mathcal{B} \;=\; \{(s, b) \in [0, \infty) \times (-\infty, +\infty) : W(s, b) \leq 0\} \,. \tag{4.6}$$

4.3 Computing $(\mathcal{Y}_Q^\dagger)^k$

Recall that $(\mathcal{Y}_Q^\dagger)^k = \bigcup_{\gamma_{i_k} \in \Gamma_k} \mathcal{Y}^\dagger_{Q(\gamma_{i_k})}$, where Γ_k is the finite discrete set of sampled γ values at the sampling discretization level k. For a given $\gamma = \gamma_{i_k} \in \Gamma_k$, the MV point $\mathcal{Y}^\dagger_{Q(\gamma_{i_k})}$ in the computed MV embedded objective set $(\mathcal{Y}_Q^\dagger)^k$ of the above-described asset-liability problem can be determined as described below.

Let $\tau = T - t$, and $\tau_j = T - t_i$, $i = 0, \ldots, M$, $j = M - i$, be the time to maturity at the ith event time. Here, $\tau_0 = T$ and $\tau_M = 0$. In addition, for use later in the paper, we denote by $\tau_j^+ = \tau_j + \epsilon$, where $\epsilon \to 0^+$. We denote by $\bar{a}_j$, $k = 0, \ldots, M-1$, the withdrawal amount in terms of the backward time variable τ. Then, we have

$$\bar{a}_j = a(\tau_{j+1} - \tau_j)e^{f(T - \tau_j)}, \quad j = 0, \ldots, M-1.$$

We further denote by c_j, $j = 0, \ldots, M-1$, the control variable representing the amount of the risk-free asset after the rebalancing of the portfolio at the event time τ_j has been carried out; c_j can take any value in $\mathcal{Z} = (-\infty, +\infty)$.

4.3.1 Value function

Define the value function $V(s, b, \tau)$ as

$$V(s,b,\tau) = \inf_{c(\cdot)} \left\{ E^{x,t}_{c(\cdot)} \left[(W_T - \frac{\gamma}{2})^2 \right] \right\} , \tag{4.7}$$

which, apart from the constant factor $\gamma^2/4$, is the objective function in equation (2.8). In addition, we define the following operators

$$\begin{aligned} \mathcal{L}V &\equiv \frac{\sigma^2 s^2}{2} V_{ss} + (\eta - \lambda\kappa) s V_s + rbV_b - \lambda V \ , \\ \mathcal{J}V &\equiv \int_0^\infty p(\xi) V(\xi s, b, \tau) \ d\xi \ . \end{aligned} \tag{4.8}$$

Rebalancing/liquidation conditions and an associated PIDE. At time $\tau = \tau_j$, $j = 0, \ldots, M-1$, we enforce the following conditions:

(1) If $(s, b) \in \mathcal{B}$, we enforce the liquidation condition

$$V(s, b, \tau_j^+) = V(0, W(s,b) - \bar{a}_j, \tau_j) \ . \tag{4.9}$$

(2) If $(s, b) \in \mathcal{N}$, we enforce the rebalancing optimality condition

$$\begin{aligned} V(s, b, \tau_j^+) &= \min_{c_j \in \mathcal{Z}} V(S^+, B^+, \tau_j) \\ S^+ &= s + b - \bar{a}_j - c_j \ \ ; \ \ B^+ = c_j \end{aligned} \tag{4.10}$$

subject to the leverage condition

$$\frac{S^+}{S^+ + B^+} \ \le q_{\max} \tag{4.11}$$

where $q_{\max}$ is a known constant with a typical value in $[1.5, 2.0]$. Note that, for the special case of τ_0, we have $V(s, b, \tau_0) = \Big(W(s,b) - \dfrac{\gamma}{2} \Big)^2$.

Within each time period $(\tau_j^+, \tau_{j+1}]$, $j = 0, \ldots, M-1$, we have

(1) If $(s, b) \in \mathcal{B}$, we enforce the liquidation condition

$$V(s, b, \tau) = V(0, W(s,b), \tau) \ . \tag{4.12}$$

(2) If $(s, b) \in \mathcal{N}$, $V(s, b, \tau)$ satisfies the Partial Integro-Differential Equation (PIDE)

$$V_\tau = \mathcal{L}V + \mathcal{J}V, \tag{4.13}$$

subject to the initial condition (4.10).

Localization. The domain for conditions (4.9)-(4.12) and the PIDE (4.13) is $(s, b, \tau) \in \Omega_j^{\infty} \equiv [0, \infty) \times (-\infty, +\infty) \times [\tau_j^+, \tau_{j+1}]$. For computational purposes, we localize this domain to the set of points

$$(s, b, \tau) \in \Omega_j \quad = \quad [0, s_{\max}) \times [-b_{\max}, b_{\max}] \times [\tau_j^+, \tau_{j+1}] \ , \tag{4.14}$$

where $s_{\max}$ and $b_{\max}$ are sufficiently large positive numbers (and are the same for all event time periods). Let $s^* < s_{\max}$. Following [13], we define the following computational sub-domains:

$$\begin{aligned}
\Omega_{s_0} &= \{0\} \times [-b_{\max}, b_{\max}] \times [\tau_j^+, \tau_{j+1}], \ \Omega_{s^*} = (s^*, s_{\max}] \times [-b_{\max}, b_{\max}] \times [\tau_j^+, \tau_{j+1}], \\
\Omega_{\mathcal{B}} &= \{(s, b, \tau) \in \Omega_j \backslash \Omega_{s_0} \backslash \Omega_{s^*} \ : W(s, b) \leq 0\}, \quad \Omega_{in} = \Omega_j \backslash \Omega_{s_0} \backslash \Omega_{s^*} \backslash \Omega_{\mathcal{B}}, \\
\Omega_{b_{\max}} &= (0, s^*] \times [-b_{\max} e^{rT}, -b_{\max}) \cup (b_{\max}, b_{\max} e^{rT}] \times [\tau_j^+, \tau_{j+1}].
\end{aligned}$$

At time $\tau = \tau_j$, we enforce (i) the liquidation condition (4.9) in $\Omega_{\mathcal{B}}$, and (ii) the optimality condition (4.10) in Ω_{in}. Within each time period $(\tau_j^+, \tau_{j+1}]$, $j = 0, \ldots, M-1$, we have the following localized problem:

$$\begin{aligned}
&V_\tau = rbV_b, \quad (s, b, \tau) \in \Omega_{s_0}; \\
&V_\tau = (\sigma^2 + 2\eta + \lambda\kappa_2)V, \quad (s, b, \tau) \in \Omega_{s^*}, \ \text{where } \kappa_2 = E[(J-1)^2]; \\
&V = V(0, W(s, b), \tau), \quad (s, b, \tau) \in \Omega_{\mathcal{B}}; \\
&V_\tau = \mathcal{L}V + \mathcal{J}_\ell V, \quad (s, b, \tau) \in \Omega_{in}, \ \text{where } \mathcal{J}_\ell V = \int_0^{s_{\max}/s} p(\xi) V(\xi s, b, \tau) \ d\xi; \\
&V = \left(\frac{b}{b_{\max}}\right)^2 V(s, \text{sgn}(b) b_{\max}, \tau), \quad (s, b, \tau) \in \Omega_{b_{\max}} \ .
\end{aligned} \tag{4.15}$$

Some guidelines for choosing $s^*, s_{\max}$ which minimize the effect of the localization error for the jump terms can be found in [16]. We refer the reader to [13] for relevant details regarding a derivation of (4.15).

We numerically solve the localized problem (4.15) using finite differences with a semi-Lagrangian timestepping method as described in [13].

4.3.2 Expected value problem.

We denote by $c_\gamma^*(\cdot)$ the optimal control of problem (4.7). Once we have determined $c_\gamma^*(\cdot)$, we use this control to determine

$$E_{c_\gamma^*}^{x,t}[W_T] \ , \tag{4.16}$$

since this information is needed in order to determine the corresponding MV embedded point. This step essentially involves solving an associated linear PIDE over each event time period $[\tau_j, \tau_{j+1}]$, $k = 0, \ldots, M-1$, details of which are similar to those described in [13], and hence, are omitted.

Using numerical solutions for equations (4.7) and (4.16) at the event time $\tau_M = t_0$, we then compute the embedded MV point

$$\mathcal{Y}_{Q(\gamma)}^{\dagger} \equiv \{(\mathcal{V}_\gamma^*, \mathcal{E}_\gamma^*)\} = \left\{ \left(Var_{c_\gamma^*(\cdot)}^{x_0,0}[W_T], E_{c_\gamma^*(\cdot)}^{x_0,0}[W_T] \right) \right\}. \tag{4.17}$$

Repeating the above-mentioned procedure for all different values of $\gamma \in \Gamma_k$ yields the computed MV embedded set $(\mathcal{Y}_Q^{\dagger})^k$.

r	σ	η	ν	λ	ζ	$W(0)$	a	f	$q_{\max}$	T	$\Delta\tau$
.0445	0.1765	.0795	-.7883	.0585	.4505	100.	6.	0.03	1.5	20. (yrs)	1. (yr)

Table 4.1: *Parameter values for the MV asset-liability example*

Refine level (k)	Timesteps	s nodes	b nodes	γ nodes	$\gamma_{\min}$	$\gamma_{\max}$
0	30	62	30	75	-0.5×10^5	-0.5×10^5
1	60	123	59	151	-1×10^5	1×10^5
2	120	245	117	303	-2×10^5	2×10^5

Table 4.2: *Computational grid for solving the PIDE* (4.13)*. We refer the reader to [13] for more details.*

4.4 Numerical results

Recall that $(\mathcal{Y}_Q^\dagger)^k$, computed using Γ_k, is an approximation to $\mathcal{Y}_Q^\dagger$. Once we obtain $(\mathcal{Y}_Q^\dagger)^k$ for a given Γ_k, we then apply the post-processing method described in Algorithm 3.1 to $(\mathcal{Y}_Q^\dagger)^k$ to determine $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$. As shown in Section 3, if convergence occurs, this process provides an increasingly accurate estimate of $\mathcal{S}(\mathcal{Y}_Q^\dagger)$ as k increases. We illustrate this by the MV asset-liability example described in the previous section. Table 4.1 summarizes the parameter values in our example. We carry out experiments with three levels of refinement, details of which are in Table 4.2.

Remark 4.1 (Combination of refinements of Γ_k and of the PIDE grid)**.** *Suppose we denote the discretization parameter of the PIDE by h, which is inversely proportional to numbers of timesteps, s and b nodes. If we fix h, then we should observe convergence of $(\mathcal{S}(\mathcal{Y}_Q^\dagger)^k)_h$ to $(\mathcal{S}(\mathcal{Y}_Q^\dagger))_h$ as Γ_k is refined, i.e. as $k \to \infty$. However, due the finite mesh size of the PIDE grid, there is PIDE discretization error in the numerical solutions. We should then repeat the above convergence test for smaller values of h. In our experiments, we take the shortcut of combining the refinements of Γ_k and of the PIDE grid. This combination is reflected in Table 4.2.*

Remark 4.2 (Complexity)**.** *Recall that $(\mathcal{Y}_Q^\dagger)^k = \bigcup_{\gamma_{i_k} \in \Gamma_k} \mathcal{Y}^\dagger_{Q(\gamma_{i_k})}$. For a given $\gamma = \gamma_{i_k} \in \Gamma_k$, the MV point $\mathcal{Y}^\dagger_{Q(\gamma_{i_k})}$ is computed by solving the associated MV asset-liability problem as described in Subsection 4.3. Examination of the solution steps reveals that*

- *each re-balancing timestep requires solution of the local optimization problem* (4.10) *at each node.*
- *each non-rebalancing timestep requires solution of the PIDE* (4.13).

At each re-balancing times τ_j, $j = 0, \ldots, M-1$, in order to solve the local optimization problems, we discretize the control (with discretization parameter h) and the use simple linear search. We have found that using a continuous 1-D optimization method is unreliable, and often converges to a local, not global, minimum. Each optimization problem is resolved by evaluating the objective function $O(1/h)$ times. Since there are $O(1/h^2)$ nodes, and $O(1)$ re-balancing timesteps, this gives a total complexity of $O(1/h^3)$ for all re-balancing timesteps.

At each non-rebalancing timestep, the fixed-point iteration developed in [16] is used, which requires an FFT at each iteration. The total complexity at each non-rebalancing timestep is then $O(1/h^2|\log h|)$, which amounts to a total complexity of $O(1/h^3|\log h|)$ for all non-rebalancing timesteps. Thus, for a single γ, the total complexity is $O(1/h^3|\log h|)$.

Remark 4.3 (Spurious points)**.** *There is an obvious strategy which generates zero variance: invest all wealth in the risk-free asset at all withdrawal times. The certain value of $\mathcal{E}$ corresponding to this risk-free strategy, denoted by $\mathcal{E}_{rf}$, can be computed by an annuity calculation. We denote by γ_{rf} the corresponding value of γ which generates this strategy. Since this strategy has zero variance, it can be viewed as corresponding to the case $\mu \to \infty$, i.e. infinitely risk-averse. Hence, from (2.14), it follows that $\gamma_{rf} = 2\mathcal{E}_{rf}$. This value γ_{rf} should be the smallest possible value of γ which can generate a valid point in $\mathcal{S}(\mathcal{Y}_Q^\dagger)$. However, as noted in Remark 2.1, a solution to the embedded problem exists $\forall \gamma \in (-\infty, +\infty)$. Consequently, from Remark 3.2, we expect that points in $\mathcal{Y}_{Q(\gamma)}^\dagger$, $\gamma < \gamma_{rf}$, are spurious points.*

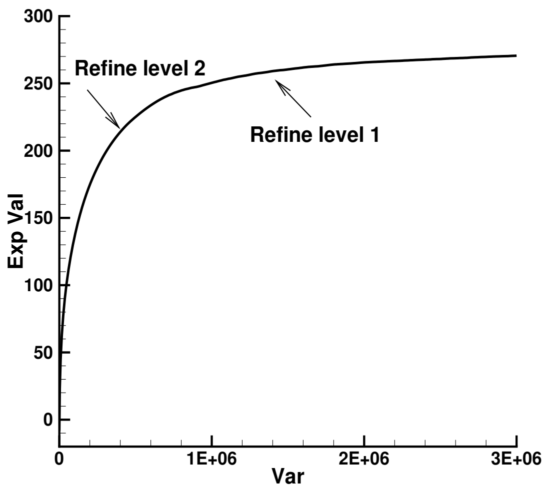
In Figure 4.1 (a), we present the computed MV embedded objective sets $(\mathcal{Y}_Q^\dagger)^k$, $k = 1, 2$, plotted as expected value versus variance. In Figure 4.1 (b), we present the same sets, but plotted as expected value versus standard deviation, which is a more practically meaningful display of the results, since standard deviation and expected value have the same units. Figure 4.1 (c) shows the same plot as in Figure 4.1 (b), but zoomed in the lower-left region, where we expect spurious points (see Remark 4.3). We make the following observations:

- In Figures 4.1 (a) and (b), the computed MV embedded objective set $(\mathcal{Y}_Q^\dagger)^k$ for $k = 2$ visually coincides with that for $k = 1$. Further refinement steps show negligible changes. This suggests convergence of the numerical solution and of the efficient frontier.
- Figure 4.1 (c) indicates that the computed MV embedded objective sets $(\mathcal{Y}_Q^\dagger)^k$, $k = 1, 2$, indeed contain spurious points in the lower-left region, as expected.

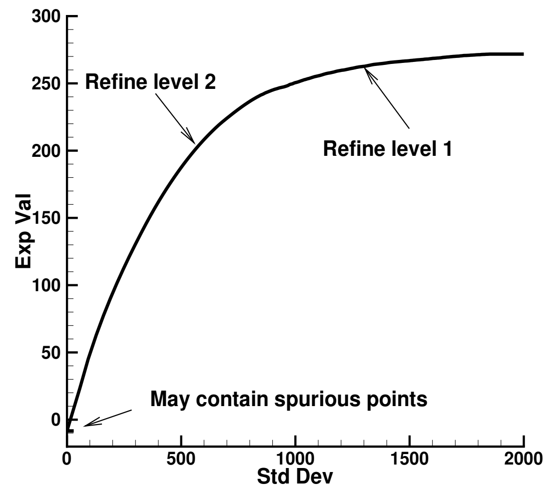
To remove spurious points in the computed MV embedded objective sets $(\mathcal{Y}_Q^\dagger)^k$, $k = 1, 2$, we apply the post-processing method described in Algorithm 3.1. After this process, we obtain the corresponding sets $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$, which are presented in Figure 4.1 (d). Again, we emphasize the strong agreement between the two levels of refinement. Based on the theoretical result of this paper and the strong agreement between $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$, $k = 1, 2$, in Figure 4.1 (d), it appears that every point in the set $\mathcal{S}((\mathcal{Y}_Q^\dagger)^2)$ is indeed close to a point in the set $\mathcal{S}(\mathcal{Y}_Q^\dagger) = \mathcal{Y}_P$, and hence, is MV scalarization optimal.

We note that, in practice, the interesting part of the efficient frontier is in the range $\gamma \in [\gamma_{rf}, q\, |\gamma_{rf}|]$, with q being problem dependent, and γ_{rf}, as mentioned in Remark 4.3, is the smallest γ which can generate a valid point in $\mathcal{S}(\mathcal{Y}_Q^\dagger)$. In our case, $q = 10$ proved to be a reasonable parameter. With this range of values for γ, the convergence of $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$, similar to what we observed in Figure (4.1), can be obtained with only 15 γ points for $k = 1$, and 29 γ points for $k = 2$.

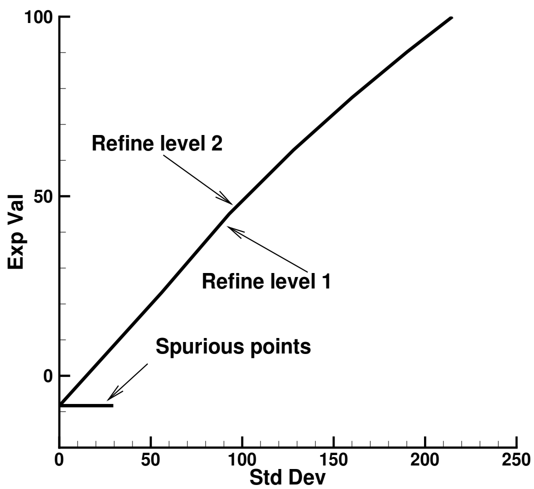
Theorem 3.1 states that, for fixed μ, the limit points of $\mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)$, $k \to \infty$, are points in $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger))$. However, the convergence of this procedure will likely be highly dependent on μ. For example, if μ is small (i.e. small slope of the supporting hyperplane), then the numerical results will be very sensitive to small errors. To illustrate this effect, we carried out the following experiments.



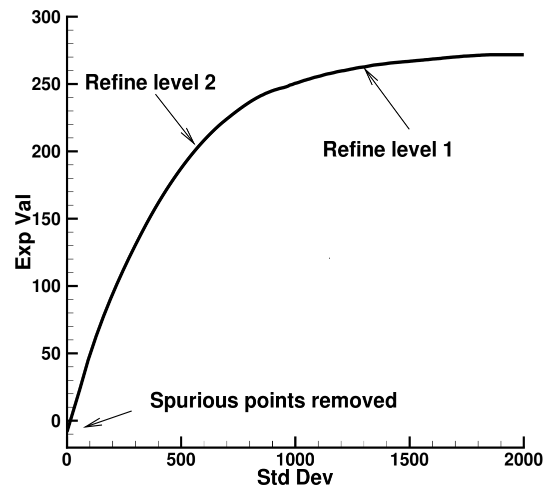
(a) $(\mathcal{Y}_Q^\dagger)^k$, $k = 1, 2$ *(Expected value vs. variance)*



(b) $(\mathcal{Y}_Q^\dagger)^k$, $k = 1, 2$ *(Expected value vs. standard deviation)*



(c) $(\mathcal{Y}_Q^\dagger)^k$, $k = 1, 2$ *(Expected value vs. standard deviation – zoomed in)*



(d) $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$, $k = 1, 2$ *(Expected value vs. standard deviation)*

FIGURE 4.1: *Plot of* $(\mathcal{Y}_Q^\dagger)^k$ *and* $\mathcal{S}((\mathcal{Y}_Q^\dagger)^k)$, $k = 1, 2$, *of the MV asset-liability example with parameters in Table 4.1.*

First, we compute $(\mathcal{Y}_Q^\dagger)^k, k = 0, 1, 2$. We then selected fixed values of μ, which did not correspond to any of the values of $\gamma \in \Gamma_k$. Then, for the fixed values of μ, we compute $\mathcal{S}_\mu((\mathcal{Y}_Q^\dagger)^k)$. Since we have only a finite number of points in $(\mathcal{Y}_Q^\dagger)^k$, then this is easily done by exhaustive search. If $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)^k$ is not a singleton, then we pick that element of $\mathcal{S}_\mu(\mathcal{Y}_Q^\dagger)^k$ which has the smallest variance

	$\mu = 0.1$		$\mu = 0.3$		$\mu = 0.4$	
k	$\sqrt{\mathcal{V}_k^{\min}}$	$\mathcal{E}_k^{\min}$	$\sqrt{\mathcal{V}_k^{\min}}$	$\mathcal{E}_k^{\min}$	$\sqrt{\mathcal{V}_k^{\min}}$	$\mathcal{E}_k^{\min}$
0	711.29	217.87	326.08	129.17	326.08	129.17
1	832.71	238.90	307.60	130.26	264.69	114.55
2	911.84	250.71	298.10	130.90	247.01	111.91

TABLE 4.3: *Numerical illustration of Theorem 3.1.*

and expectation, denoted by $(\mathcal{V}_k^{\min}, \mathcal{E}_k^{\min})$. In Table 4.3, we present $(\sqrt{\mathcal{V}_k^{\min}}, \mathcal{E}_k^{\min})$ for $k = 0, 1$, and 2. Observe that, as expected, for moderate values of $\mu = 0.3, 0.4$, the values of mean and standard deviation appear to converge somewhat faster than for the $\mu = 0.1$ case. However, from a practical point of view, we can see that these errors for small μ have very little effect on the efficient frontier (this corresponds to large variances), as can be seen in Figure 4.1 (a). In general, there does not seem to be a consistent order of rate of convergence.

5 Application to higher dimensional problems

Our main result, Theorem 3.1, is concerned with the convergence of the discretely sampled solution of equation (2.8). This result is independent of any particular numerical technique used to solve the control problem (2.8).

However, it is of practical interest to solve the asset-liability problem with several risky assets. A difficulty in this case case is that in order to ensure convergence to the viscosity solution of the optimal control HJB equation (4.13), we need to construct monotone discretization schemes [3]. For the case of correlated risky assets, construction of such schemes is a matter of on-going research. We refer the reader to [14] for a discussion of the *wide stencil* approach to this problem.

In addition, of course, solving the HJB PDE in higher dimensions becomes problematic, due to the computational complexity. Suppose there is one risk-free asset and d risky assets. Let the discretization parameter for the PIDE be h, as discussed in Remark 4.2. For simplicity, we consider the case where the risky assets follow a pure diffusion process (no jumps). Then, using an argument similiar to that used in Remark 4.2, we find that the complexity for a solving the HJB equation for a single value of γ is $O(1/h^{d+2})$, which increases rapidly as d increases.

If x is the state vector of the system, then recall that our objective is to find the control $c(\cdot)$ which solves

$$V(x, \tau) = \inf_{c(\cdot)} \left\{ E_{c(\cdot)}^{x,t} \left[(W_T - \frac{\gamma}{2})^2 \right] \right\} . \tag{5.1}$$

An alternative numerical approach for determining the solution of equation (5.1) would be use a Monte Carlo method. This would require formulating the control problem (5.1) as a system of Backward Stochastic Differential Equations (BSDEs). Some promising results for Monte Carlo methods using BSDEs in the context of control problems have been obtained recently, see for example [6, 18, 26]. Theorem 3.1 can then be used when sampling the solution of equation (5.1) for a finite number of values of γ. Theorem 3.1 ensures us that as the sampling mesh becomes finer, the results of these Monte Carlo computations generate an accurate approximation to the true efficient frontiers.

6 Conclusion

Many optimal stochastic control problems in finance can be posed in term of a continuous time MV optimization problem, which involves two conflicting objectives. Using the standard scalarization technique, this multi-criteria optimization problem can be reformulated as a single-objective MV scalarization optimization problem. The goal is to determine the original MV scalarization optimal set $\mathcal{Y}_P$. However, dynamic programming can not be applied to the above scalarization optimization problem, due to the presence of the variance term. To overcome this difficulty, the embedding technique of [21, 33] can be applied to determine the set of computed MV embedded objectives $\mathcal{Y}_Q^\dagger$, which, in general, is a superset of the original MV scalarization optimal set $\mathcal{Y}_P$. As a result, the MV efficient frontiers generated by the embedding technique may contain spurious points, which do not belong to the original MV scalarization optimal set $\mathcal{Y}_P$.

In [27], it is established that spurious points in the computed MV embedded objective set $\mathcal{Y}_Q^\dagger$ are those which are not MV scalarization optimal with respect to $\mathcal{Y}_Q^\dagger$. In addition, it is established that the set of MV SOPs with respect to the computed MV embedded objective set $\mathcal{Y}_Q^\dagger$ is identical to the original MV scalarization optimal set $\mathcal{Y}_P$. Based on these two results, a simple, yet effective, post processing technique is developed to eliminate spurious points in the computed MV embedded objective set $\mathcal{Y}_Q^\dagger$.

In the context of numerical computation, however, significant complexities remain, since it is only possible to solve the embedded problem for a finite number of values of the embedding parameter, and hence we can only obtain a finite subset of the computed MV embedded objective set $\mathcal{Y}_Q^\dagger$. An important question is whether or not, for sufficiently large number of sampling points of the embedding parameter, the set of SOPs with respect to the afore-mentioned finite subset of $\mathcal{Y}_Q^\dagger$ can sufficiently well approximate the set of SOPs with respect to $\mathcal{Y}_Q^\dagger$.

In this paper, we establish that, for sufficiently large number of sampling points of the embedding parameter, every limit point in the set of SOPs with respect to the computed finite subset of $\mathcal{Y}_Q^\dagger$ is a point in the set of SOPs with respect to $\mathcal{Y}_Q^\dagger$, and hence, is MV scalarization optimal. This result combined with the analysis and post-processing numerical method developed in [27] form a practical numerical framework for eliminating spurious points from the computed MV embedded objective set. This framework can essentially be viewed as complementing the theoretical results of the popular embedding technique developed in [21, 33] for continuous time (or multi-period) MV optimization.

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