Convertible Bonds with Call Notice Periods

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Abstract

In practice, convertible bonds can often be called only if notice is given to the holders. Most methods for valuing convertible bonds assume that the bond is continuously callable. In this paper, we develop an accurate PDE method for valuing convertible bonds with a finite notice period. Example computations are presented which illustrate the effect of varying notice periods. The results are compared with a recently published approximation method.

1   Introduction

Convertible bonds (or convertibles) have become an important instrument in the financial markets. Having properties of both stocks and bonds, convertible bonds can be an attractive alternative for investors. Studies suggest that the average return of convertible bonds in the last few years were as high as the returns of the stock market, although they incorporate a lower risk [vdHKL02, LR93]. There are different reasons for a company to issue convertible bonds. Tax considerations in some countries lead to an advantage in issuing convertibles instead of bonds. Another possibility is that a small, fast growing company needs a debt but has poor credit rating.

The convertible bond market is not as standardized as the exchange traded stock market. Convertibles can incorporate a variety of features. The instrument might be convertible into shares of the issuing company or in some cases into shares of a different company. Usually convertibles may be converted by the holder at any time. Often, these bonds can be put to the issuer at specific dates for a guaranteed price. In addition, the issuer may have the right to redeem the convertible at a call price or force a conversion into stocks. To keep the convertibles attractive in this case, so called soft and hard call constraints are devised. The hard call constraint prohibits a forced conversion in the initial life of the contract. The soft call constraint can define a notice period before a forced conversion can take place. As well, the stock may have to be above a trigger price for a specified time before a call can take place.

Many authors have discussed the delayed call phenomena [LK03, GKK02, AB02, AKW01]. It seems that companies tend to call convertibles nonoptimally. The observed stock price at which corporations issue a call notice is often well above the stock price which is optimal assuming

the validity of the Ingersoll result [Ing77a]. Different explanations for this behavior have been proposed including tax considerations and a preference for conversion into stock instead of leaving the bond as a liability. Other authors suggest that the call notice period is not taken into account properly. Empirical studies with such a model suggest that the notice period is indeed a possible part of an explanation [Asq95].

Lau and Kwok [LK03] present a detailed lattice method for convertibles with notice periods. Their results are similar to our findings but a precise implementation of a PDE method reveals more details of the optimal call strategy. Further, the PDE method can use different techniques to increase the rate of convergence for accurate solutions and a concise implementation of all cash flows is possible.

In the following, a one factor model for convertible bonds is presented. The optimal call and conversion strategy are determined by the PDE solution. These strategies are compared with suboptimal approximate methods.

This work is organized as follows: we present the standard model for convertible bonds with credit risk and we provide a short summary of new developments in this area. We derive the equations which take into account, in a rigorous manner, the call notice period. An outline for the numerical algorithm is presented, followed by a case study. Previously published approximations for the optimal call policy are revisited and compared with results from our new model. Finally we conclude and summarize.

2 Models for convertible bonds

Our main focus here is on modelling the call notice period. We will restrict attention to the case where interest rates are deterministic. This is in line with current practice since it is commonly believed that the effect of stochastic interest rates on convertible pricing and hedging is a small effect, compared to stochastic stock prices. Dilution effects will also be ignored in the following.

2.1 No default risk

For ease of explanation, consider first the case where we ignore the credit risk of the issuer of the convertible. We will assume that that the stock price S evolve according to the process

$$dS = \mu S dt + \sigma S dZ \tag{2.1}$$

where μ is the drift rate, σ is the volatility of S and dZ is the increment of a Wiener process, then, following the standard arguments, we get for the value of any contingent claim on S, denoted by V satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.2}$$

Consider the case of a convertible bond which has no put or call provisions, and can only be converted at the terminal time T. If the convertible has a face value F, and can be converted into

κ shares, then the value of the convertible V is given from the solution to equation (2.2), with the terminal condition

$$V(S, t = T) = \max(F, \kappa S) .$$ (2.3)

2.2 Call and Put Provisions and cash flows

Assume that the convertible is continuously callable at call price $B_c(t)$ and can be converted by the holder into the put price $B_p(t)$ or shares worth κS. Then, the pricing problem can be stated as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \;\geq\; 0$$ (2.4)

$$V(S,t) \;\geq\; \max(B_p(t), \kappa S)$$ (2.5)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \;\leq\; 0$$ (2.6)

$$V(S,t) \;\leq\; \max(B_c(t), \kappa S)$$ (2.7)

where at least one of equations (2.4)-(2.5) or (2.6)-(2.7) holds, and at least one of the inequalities holds with equality at each point in the solution domain.

If a discrete dividend D is paid at time t_d, then the usual no-arbitrage arguments imply that

$$V(S - D, t_d^+) = V(S, t_d^-).$$ (2.8)

where t_d^- is the time immediately before the dividend payment, and t_d^+ is the time immediately after the payment.

Consider coupon payments c_i paid at times $t_{c,i}$. Denote the time immediately before the payment as $t_{c,i}^-$ and immediately after the coupon payment as $t_{c,i}^+$. The price of the convertible then drops according to

$$V(S, t_{c,i}^+) = V(S, t_{c,i}^-) - c_i.$$ (2.9)

2.3 Credit Risk

The above model ignores the credit risk of the issuer of the bond. Clearly, this is an important effect.

2.3.1 Credit Risk: The T&F model

Tsiveriotis and Fernandes [TF98] proposed a model whereby the option component of the convertible was discounted at the risk-free rate, and the bond component was discounted at a risky rate. Let the spread s between a risk-free bond and a risky bond be given by

$$s = (1 - R)p(S,t)$$ (2.10)

where R is the recovery rate, and $p(S,t)$ is a function which can be calibrated to market data.

Under the *T&F* model, the value of the convertible is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - r(V - B) - sB \;\geq\; 0 \tag{2.11}$$

$$V(S,t) \;\geq\; \max(B_p(t), \kappa S) \tag{2.12}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - r(V - B) - sB \;\leq\; 0 \tag{2.13}$$

$$V(S,t) \;\leq\; \max(B_c(t), \kappa S) \tag{2.14}$$

where at least one of equations (2.11)-(2.12) or (2.13)-(2.14) holds, and at least one of the inequalities holds with equality at each point in the solution domain. The bond component B in equations (2.11)-(2.14) is given from the solution to

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} + rS\frac{\partial B}{\partial S} - sB = 0 \tag{2.15}$$

subject to the boundary conditions

$$\begin{aligned}
B &= 0 \;;\quad \text{if } V = \max(B_c, \kappa S) \\
B &= V \;;\quad \text{if } V = B_p
\end{aligned} \tag{2.16}$$

with terminal conditions

$$\begin{aligned}
V(S,T) &= \max(F, \kappa S) \\
B(S,T) &= F \;;\quad F > \kappa S \\
&= 0 \;;\quad F \leq \kappa S
\end{aligned} \tag{2.17}$$

2.3.2 Credit Risk: The AFV model

The T&F model was derived in a very heuristic manner, and, as pointed out in [AFV02], seems to be inconsistent in some cases. Ayache, Vetzal and Forsyth derive a different model, based on a hedging portfolio where the risk due to the normal diffusion process is eliminated, and assuming a Poisson default process. [AFV02]. The probability of default in $[t, t + dt]$, conditional on no-default in $[0,t]$ is $p(S,t)$.

This model allows different scenarios in the case of default. Upon default, it is assumed that the stock price jumps according

$$S^+ = S^-(1 - \eta), \;\; 0 \leq \eta \leq 1 \tag{2.18}$$

where S^+ is the stock price after default, and S^- is the stock price just before default. Further, the holder of the convertible can choose upon default between:

1. Recovering RX, where $0 \leq R \leq 1$ is the recovery factor. There are various possible assumptions for X, e.g. face value of bond, discounted bond cash flows, or pre-default value of the bond component of the convertible,

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2. shares worth $\kappa S(1 - \eta)$.

For simplicity in the following, we will assume that the recovery rate $R = 0$.

This leads to the following partial differential inequality for the convertible value V [AFV02]

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r + p\eta)S\frac{\partial V}{\partial S} - (r+p)V + p\kappa S(1-\eta) \geq 0 \tag{2.19}$$

$$V(S,t) \geq \max(B_p(t), \kappa S) \tag{2.20}$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r + p\eta)S\frac{\partial V}{\partial S} - (r+p)V + p\kappa S(1-\eta) \leq 0 \tag{2.21}$$

$$V(S,t) \leq \max(B_c(t), \kappa S) \tag{2.22}$$

where, as for the *T&F* model, either one of (2.19)-(2.20) or (2.21)-(2.22) hold, and one of the inequalities holds with equality at each point in the solution domain. The terminal condition is given in equation (2.3).

3   Notice periods

To make a convertible bond more attractive for investors, there are usually constraints on the call provision. A common feature is a call with a notice period. If the issuer wants to call the convertible and force a conversion, he has to notify the holder. The holder then has T_n time to decide to take the face value or convert into shares. So, the issuer is effectively giving the holder a put option on his shares plus the shares themselves. The longer the notice period, the more valuable is this put option.

3.1   A Model for the valuation of CBs with a notice period

The value of the shares plus the put option can be described as the forward price $V_{c,t}$ of a new convertible bond with maturity $t + T_n$, and terminal value

$$V_{c,t}(S,t+T_n) = \max(B_c(t+T_n), \kappa S) \tag{3.1}$$

Note that the call value B_c includes accrued interest. Based on the assumption that the issuer wants to minimize the value of outstanding convertible bonds, he has to minimize the market value of the convertible [Ing77a]. So, the issuer will call the convertible as soon as the forward price $V_{c,t}$ exceeds the price of the convertible. That means that in the model for convertibles (T&F or AFV) we need to replace all conditions with a call price B_c by a condition with the forward price $V_{c,t}$.

In the *T&F* case, equations (2.11)-(2.14) become

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - r(V - B) - sB \geq 0 \tag{3.2}$$

$$V(S,t) \geq \max(B_p(t), \kappa S) \tag{3.3}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - r(V - B) - sB \leq 0 \tag{3.4}$$

$$V(S,t) \leq V_{c,t}(S, \hat{t} = t) \tag{3.5}$$

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where the bond component B is given from the solution to

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} + rS\frac{\partial B}{\partial S} - sB = 0 \tag{3.6}$$

subject to the boundary conditions

$$\begin{aligned} B &= 0 \; ; \quad \text{if } V = \max(V_{c,t}, \kappa S) \\ B &= V \; ; \quad \text{if } V = B_p \end{aligned} \tag{3.7}$$

with terminal conditions

$$\begin{aligned} V(S,T) &= \max(F, \kappa S) \\ B(S,T) &= F \; ; \; F > \kappa S \\ &= 0 \; ; \; F \le \kappa S \end{aligned} \tag{3.8}$$

$V_{c,t}(S,\hat{t})$ satisfies

$$\frac{\partial V_{c,t}}{\partial \hat{t}} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{c,t}}{\partial S^2} + rS\frac{\partial V_{c,t}}{\partial S} - r(V_{c,t} - \hat{B}) - s\hat{B} \ge 0$$
$$V_{c,t}(S,\hat{t}) \ge \max(B_p, \kappa S) \tag{3.9}$$

with terminal condition

$$V_{c,t}(S,\hat{t} = t + T_n) = \max(B_c(t+T_n), \kappa S) \tag{3.10}$$

and $\hat{B}(S,\hat{t})$ satisfies

$$\frac{\partial \hat{B}}{\partial \hat{t}} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{B}}{\partial S^2} + rS\frac{\partial \hat{B}}{\partial S} - s\hat{B} = 0 \tag{3.11}$$

with terminal conditions

$$\begin{aligned} \hat{B}(S,\hat{t} = t + T_n) &= B_c(t+T_n) \; ; \; B_c > \kappa S \\ &= 0 \; ; \; B_c \le \kappa S \end{aligned} \tag{3.12}$$

For the AFV model the following equations need to be solved for

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r+p\eta)S\frac{\partial V}{\partial S} - (r+p)V + p\kappa S(1-\eta) \ge 0 \tag{3.13}$$
$$V(S,t) \ge \max(B_p(t), \kappa S) \tag{3.14}$$
$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r+p\eta)S\frac{\partial V}{\partial S} - (r+p)V + p\kappa S(1-\eta) \le 0 \tag{3.15}$$
$$V(S,t) \le V_{c,t}(S,\hat{t} = t) \tag{3.16}$$

with $V_{c,t}(S,\hat{t})$ satisfying

$$\frac{\partial V_{c,t}}{\partial \hat{t}} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V_{c,t}}{\partial S^2} + (r+p\eta)S\frac{\partial V_{c,t}}{\partial S} - (r+p)V_{c,t} + p\kappa S(1-\eta) \ge 0 \tag{3.17}$$
$$V_{c,t}(S,\hat{t}) \ge \max(B_p, \kappa S) \tag{3.18}$$

with terminal condition

$$V_{c,t}(S,\hat{t} = t + T_n) = \max(B_c(t+T_n), \kappa S) \tag{3.19}$$

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4 Numerical Algorithm

The PDEs in the T&F and the AFV model are parabolic partial differential equations, similar to the Black-Scholes equation which can only be solved analytically for special cases. In this general setting with the inequality constraints, an analytical solution is not possible. However, it is possible to solve the equations numerically.

In this paper, the solution of the PDEs in the T&F as well as the AFV case are computed via a discretization in two dimensions: S and t. The solution is generated at discrete values $V(S_i, t_n) = V_i^n$, $S = S_1, \ldots, S_{imax}$. As usual, the solution proceeds backwards in time. Given the terminal (payoff) conditions at $t_n = T$, the solution at t_{n-1} is generated using an implicit finite difference scheme. Dividend and coupon payments are included as in equation (2.8)-(2.9).

The pseudo code in Listing 1 illustrates the solution process. We assume the existence of a function `discrete_timestep` which, given $\mathcal{V}(t_n) = V_1^n, \ldots, V_{imax}^n$, does one time step of the implicit solution method to return $\mathcal{V}(t_{n-1}) = V_1^{n-1}, \ldots, V_{imax}^{n-1}$.

An important detail in this implementation is the treatment of cash flows which occur within the notice period. There are usually no details written in the convertible bond contracts about what happens if the issuer calls and there is a coupon payment within the notice period. So, we assume that there is no special treatment in this case and the coupon will be paid as usual. A similar reasoning is valid for dividends. Both cash flows, coupon and dividend, which are paid at time t_i are applied at $t = t_i$ to calculate $V(t, S)$ and at $\hat{t} = t_i$ to calculate the value for the constraint $V_{c,t}(S, \hat{t})$. This allows the holder to obtain the coupon after a call notice and then convert into share before the end of the notice period to get the dividend. The algorithm in Listing 1 can be easily adapted for a different treatment of these cash flows.

5 Case Study

The call price B_c and the put price B_p in the previous equations include accrued interest. Specifically, let B_{cl}, B_{pl} be the clean call and put prices. The actual call price is computed by

$$B_c(t) = B_{cl}(t) + A(t), \tag{5.1}$$

where $A(t)$ is the accrued interest, a fraction of the next coupon payment. If the last payment was at t_{i-1} and the next payment worth c_i is paid at t_i, then be accrued interest $A(t)$ is

$$A(t) = \frac{t_i - t}{t_i - t_{i-1}} c_i. \tag{5.2}$$

In order to obtain comparable results for both T&F and AFV methods, we set $R = 0$ and assume $\eta = 1$ (stock jumps to zero on default). Consequently in the T&F case, $s = p$. For the hazard rate p, we use the model suggested by Muromachi [Mur99]

$$p(S) = p_0 \left(\frac{S}{S_0} \right)^\alpha. \tag{5.3}$$

The parameters p_0, α can be calibrated to market data. In the following, we use $p_0 = 0.02$, $\alpha = -1.2$ and $S_0 = 100$, which are typical parameters found in market data [Mur99].

Listing 1: *Pseudo code for the numerical algorithm*

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function vector= discrete_timestep ($\mathcal{V}_{\text{old}}$, $\mathcal{S}$, t , constraint , ...)
  This function is a discrete version of the T&F or the AFV model.
  It uses e.g. an implicit method to compute the values $\mathcal{V}(t - \Delta t)$ from
  \\$\mathcal{V}(t)$ *and returns the result as a vector. The " constraint " on the values*
  \\$\mathcal{V}$ *is implicitly applied e.g. with a penalty method [FV02].*

function vector= convertible_with_notice ($\mathcal{V}_{\text{terminal}}$, $\mathcal{S}$, T , σ , r , ...)
{
  Computes the values of a convertible with a notice period
  and returns the prices $\mathcal{V}(S_i) \forall i$ *at* $t = 0$ *as a vector*

  $\mathcal{V} = \mathcal{V}_{\text{terminal}}$;
  for all timesteps from $t = T$ down to $t = 0$
  {
    if notice to call possible
    {*solve for the constraint*
      $B_c = B_{cl}$ + accrued_interest   at $t + T_n$;
      $\mathcal{V}_{c,t}(S_i) = \max(B_c, \kappa S_i) \forall i$;*the terminal condition*
      for all timesteps from $\hat{t} = t + T_n$ down to $\hat{t} = t$
      {
        constraint $= \{\mathcal{V}_{c,t}(S_i) \geq \max(B_p(\hat{t}), \kappa S_i) \forall i\}$;
        $\mathcal{V}_{c,t}$ = discrete_timestep ($\mathcal{V}_{c,t}$, $\mathcal{S}$, $\hat{t}$, constraint , ...);

        if cash flow occurs between last timestep and $\hat{t}$
          apply_cash_flow ();
      }*end of inner time-stepping for loop*
    }*end of constraint block*

    constraint $= \{(\mathcal{V}(S_i) \geq \max(B_p(t), \kappa S_i)) \wedge (\mathcal{V}(S_i) \leq \max(\mathcal{V}_{c,t}(S_i), \kappa \, S_i)) \forall i\}$;
    $\mathcal{V}$ = discrete_timestep ($\mathcal{V}$, $\mathcal{S}$, t , constraint , ...);

    if cash flow occurs between last timestep and t
      apply_cash_flow ();
  }*end of time-stepping for loop*
  return $\mathcal{V}$;
}*end of function convertible_with_notice*

5.1 Example Data

The base case data is given in Table 5.1.

Table 5.1 *Specifications of a convertible bond.*

general features	
Conversion ratio κ	1
Face value F	100
Coupon payment c_i	2, semi annually (4% per annum)
Maturity T	5 years
Risk free rate r	5%
Volatility σ	20%
Dividends D_i	2, paid once a year, just after the coupon
call ability	
Call period	starting after 1.0 years
Call price B_{cl}	140
Notice period T_n	1/12 years

Some of these properties will be varied so that the effect on the model price can be evaluated.

5.2 Convergence Analysis

In Table 5.2 the values are displayed for a convertible using the base case data in Table 5.1. Crank-Nickolson time stepping is used. To decrease oscillations, a method presented by Rannacher [Ran84] is used at each non-smooth initial state.

Table 5.2 shows a numerical convergence analysis. At each refinement, the number of nodes (in the S grid) and the number of time steps is doubled. The number of substeps used to determine $V_{c,t}$ (inner time stepping in loop in pseudo code, Listing 1) is also shown. For both methods, the numerical solutions appear to be converging, but the convergence rate is quite erratic. This contrasts with the smooth quadratic convergence in [FV02] for simple American options. We conjecture that the time dependent movement of the constraint $V_{c,t}$ in equation (3.5) respectively equation (3.16) may cause some difficulties in obtaining smooth convergence, as well as the effect of the accrued interest.

Each time step of the algorithm in Listing 1 requires about (#substeps+1) times the work required for a convertible bond with no notice period. In all presented cases, the constraint $V_{c,t}$ is solved on a grid with the same spacing as the grid for V. From Table 5.2, we see that a grid with 400 nodes has an error of about ± 0.02. All results in subsequent sections are reported using a 400 node or finer grid.

Table 5.2 *Convergence study with the V&F and the AFV model extended for a notice period. Substeps refer to the number of time steps used to determine $V_{c,t}$, at each discrete time.*

The T&F model

grid for V, $V_{c,t}$ $S \times t \times$ substeps	$V(s=100, t=0)$	difference	ratio
$50 \times 50 \times 1$	112.03151		
$100 \times 100 \times 2$	112.17249	0.14098	
$200 \times 200 \times 4$	112.23473	0.06224	2.27
$400 \times 400 \times 7$	112.25619	0.02146	2.90
$800 \times 800 \times 14$	112.26951	0.01332	1.61
$1600 \times 1600 \times 27$	112.27578	0.00627	2.12
$3200 \times 3200 \times 54$	112.27982	0.00404	1.55

The AFV model

grid for V, $V_{c,t}$ $S \times t \times$ substeps	$V(s=100, t=0)$	difference	ratio
$50 \times 50 \times 1$	112.4248		
$100 \times 100 \times 2$	112.5104	0.08555	
$200 \times 200 \times 4$	112.5453	0.03494	2.45
$400 \times 400 \times 7$	112.5485	0.00318	10.99
$800 \times 800 \times 14$	112.5513	0.00277	1.15
$1600 \times 1600 \times 27$	112.5504	-0.00084	-3.30
$3200 \times 3200 \times 54$	112.5508	0.00032	-2.63

5.3 Implications on the optimal call strategy

The call strategy of the issuer is an important factor for the price of the theoretical value of the convertible bond. Earlier results from Ingersoll [Ing77a], Brennan and Schwarz [BS77] state the optimal call strategy which an issuer should follow if he could call without notice. Butler [But02] extends these results for notice periods and dilution.

The optimal call strategy for a continuously callable convertible without a notice period is [Ing77a]:

> "A convertible security should be called as soon as its conversion value (i.e., the value of the common stock which would be received in the conversion exchange) rises to equal the prevailing effective call price (i.e., the stated call price plus accrued interest)... A sufficient condition for the optimal call to be exactly at the point of equality between the conversion value and the call price is that the promised coupon rate be less than the riskless rate of interest."

Or more precisely:

$$S^* = B_c \tag{5.4}$$

with B_c, being the effective call price (including the accrued) interest and S^* the optimal stock price for a call of the issuer.

This is a special case for Butler's result [But02]. His result is stated here for the case that the dilution caused by the exercise of the warrant is infinitesimally small relative to the existing equity.

Butler states that a good approximation for the optimal call strategy is to call the convertible bond if

$$\frac{S}{B_c} = e^{\sqrt{T_n}\sqrt{T}[r+\frac{\sigma^2}{2}]} \tag{5.5}$$

where B_c is the effective call price, T_n the time of the conversion period, T the maturity and r the risk free rate.

Butler's result based on the fact that the holder effectively gets the stock plus a put option upon conversion. Butler formulates the problem as

$$\min_S \{V(S,t,T) - [S(t) + P(S,t,T_n)]\}, \tag{5.6}$$

where $V(S,t,T)$ is the value of the convertible with maturity T and the stock price $S(t)$. $P(S,t,T_n)$ is the value of the put option with maturity T_n. Further, we can decompose $V(S,t,T) = B(S,t,T) + C(S,t,T)$ with B as the discounted value of the debt and C as the value of a call option with maturity T.

The minimization problem can be solved by taking the partial derivative with respect to S which gives the following first order condition:

$$\frac{\partial C(S,t,T)}{\partial S} + \frac{\partial B(S,t,T)}{\partial S} - \frac{\partial P(S,t,T_n)}{\partial S} = 1. \tag{5.7}$$

Now, Butler's result implicitly assumes the following:

- The convertible bond can only be called once, at a predetermined point in time.

- The call price is equal to the face value of the bond ($B_{cl} = F$).

- The underlying stock is not paying any dividends.

- The term structure of the risk free rate is flat.

- The default risk of the issuer does not depend on the stock price.

Using these assumptions we can conclude that $B(t)$ does not depend on S, so that $\frac{\partial B(t)}{\partial S} = 0$. Further we can use the Black-Scholes-Merton formula for European options to evaluate the partial derivative of the call and the put option. After some algebra, we get that it is optimal to call the convertible bond if the stock price reaches the level S_B^* where S_B^* is the solution to

$$\frac{\ln(S_B^*/B_c) + (r+\sigma^2/2)T_n}{\sigma\sqrt{T_n}} = \frac{\ln(S_B^*/B_c) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}. \tag{5.8}$$

This can be simplified to

$$S_B^* = B_c e^{\sqrt{T_n}\sqrt{T}[r+(\sigma^2/2)]} \tag{5.9}$$

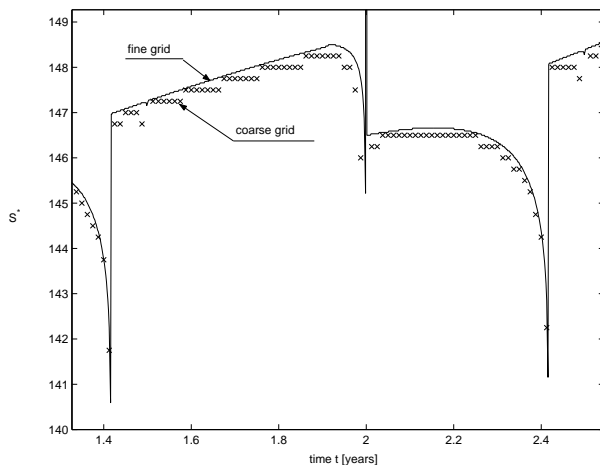which is the approximation of the optimal call policy presented by Butler [But02].

Figure 5.1: *A convertible using the data from Table 5.1: The level of stock price S^* for the optimal call strategy versus time t approximated by the AFV model using a PDE solver. The solution on a coarse grid (400 nodes in S, 400 time steps, 7 substeps) versus a fine grid (3200 nodes in S, 3200 time steps, 54 substeps) is shown.*

A more sophisticated model that takes more of the complex features of the convertible bond into account follows from the discretization of the PDE in the T&F model or the AFV model.

For each node V_i at time t_j of the discretization we check if

$$V_i(t_j) = (V_{c,t})_i(t_j) \tag{5.10}$$

or in other words: We check if the maximum constraint in equation (3.5) respectively equation (3.16) is active. At time step t_j, let $V_i(S_i, t_j)$ be the node with S_i the smallest value in $\mathcal{S}$ that results in an active constraint. Then S_i gives a good approximation for the optimal stock price level: $S^*(t_i) \approx S_i(t_j)$. This method is similar to a nearest neighbor approximation.

The error of approximating the optimal strategy by the PDE method is presented in Figure 5.1. In this Figure, S^* is approximated by a PDE solution a coarse grid and on a fine grid. The difference between the two approximations is less then 1.0.

The PDE method is capable of a precise estimate of S^* for a sufficiently fine grid. Consequently, the estimate from the PDE method can be a reference for the other approximations. In Figure 5.2, the graph "Ingersoll" shows the optimal stock price level S^* for a convertible described in Table 5.1 but without a notice period. The graph is generated by the PDE method. The value of the optimal stock price level equals to the clean call price plus the accrued interest. This was predicted by Ingersoll [Ing77a] and equation (5.4). Further studies show that the error of the PDE method is small and decreases for a finer discretization grid.

Figure 5.2 shows the convertible bond without dividends. The holder is allowed to convert any time and there are coupon payment of $c_i = 2$. In this setting, the optimal stock price for a call shows a complex behavior: The PDE method in the graph "This work" shows that there is a large drop in S^* just before a coupon payment is within the notice period and a jump as soon as the coupon payment is within the notice period. None of the other approximations shows such a
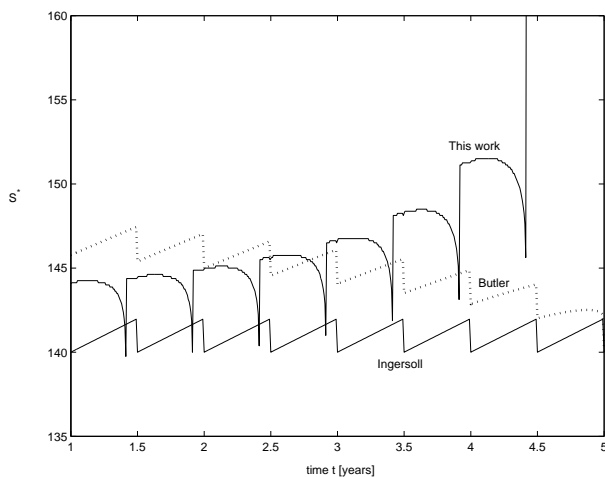
Figure 5.2: *A convertible using the data from Table 5.1 but without dividends: The level of stock price S* for the optimal call strategy over time t approximated by the AFV model using a PDE solver ($T_n = 1/12$), Butler's method ($T_n = 1/12$) and by Ingersoll $T_n = 0$ are shown.*
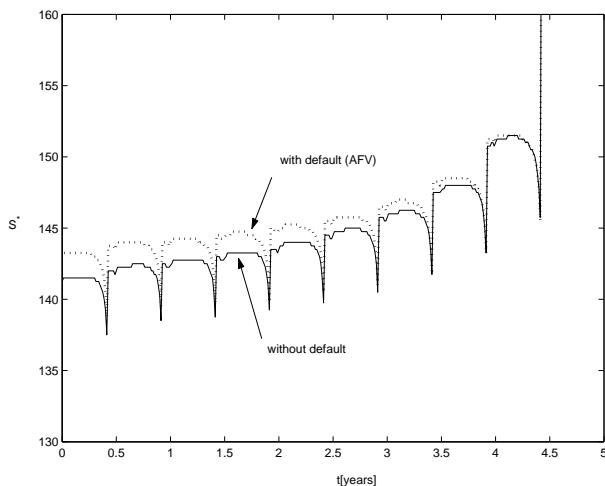

Figure 5.3: *The level S* for the optimal call strategy over time t for a convertible according to Table 5.1, but callable through the entire lifetime and no dividends are paid: The solid line is computed without a default model, the dotted line with the AFV model. The case without default has $p(S) = 0$.*

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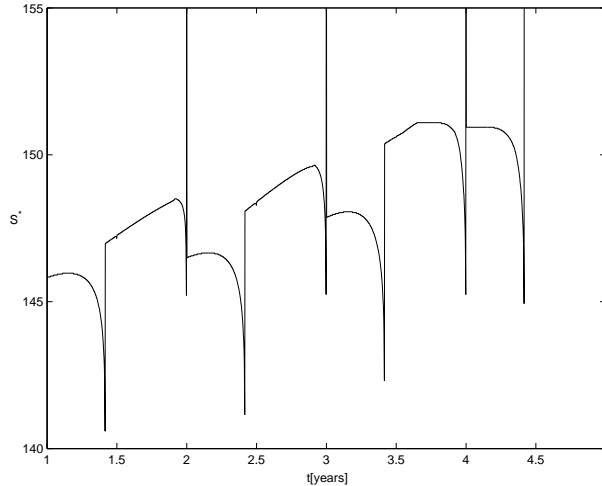
Figure 5.4: *The optimal call strategy S^* over time t computed with the extended AFV PDE method. The data is given in Table 5.1. A fine grid is used: 3200 nodes, 3200 time steps, 57 substeps.*

complex structure. The method from Butler (3) has approximately the correct value. But, instead of rising with time, it decays.

Figure 5.3 shows the effect of the default model on S^* for the AFV model. In this case, we alter the base case data in Table 5.1, so that the convertible is callable for the entire lifetime of the bond and no dividends are paid.

In the presence of dividends, the optimal conversion strategy changes, as presented in Figure 5.4. The calculations are based on the data in Table 5.1. The dividends result in the optimal stock price being higher than without dividends. Also, the dividends make calling the bond non optimal immediately prior to a dividend date ($t = 2, t = 3, t = 4$).

5.3.1 Properties of the optimal call strategy

An interesting property of the optimal call strategy (Figure 5.2) is that it does not seem to be optimal to call after the last coupon before maturity is paid. To show, why this is true, consider the issuer. He tries to minimize the value of the convertible bond. Consequently, the issuer tries to avoid the possibility that the holder gets a coupon plus the opportunity to convert into shares. Thus, the value for S^* is relatively low just before a coupon payment takes place. But, at maturity, the holder gets either the face value plus the last coupon or κ shares and no coupon. So, there is no need for the issuer to call the convertible bond because the holder cannot get both.

From Figure 5.2, we can see that in the case of a notice period, the optimal S^* at which the issuer should call the convertible is most of the time higher than B_c (line labelled Ingersoll on Figure 5.2). This is reasonable because the issuer effectively gives the holder a put on the underlying stock once the CB is called. The issuer wants to avoid having this put in the money at maturity. But, there are dips in S^* before each coupon payment, with some of them dropping below B_c.

The dips in S^* result from the treatment of coupon payments. The value of S^* drops significantly before each coupon, followed by a jump. To explain the situation, Figure 5.5 shows the

value of the convertible before and after the jump. Since the PDE is solving backwards in time, consider the values at $t = t_c - T_n + \varepsilon$: $V(t_c - T_n + \varepsilon)$ and $V_{c,t}(t_c - T_n + \varepsilon)$. These are the values of the convertible respectively of the maximum constraint. Note that there is a coupon payment within the notice period. In this situation, the coupon is paid to the holder whether or not the issuer gives a call notice. That is the reason why the value of $V_{c,t}$ is relatively high compared with the conversion price κS. So, the issuer cannot avoid paying the coupon at time t_c. The situation is different for time $t = t_c - T_n - \varepsilon$. Now, if the issuer gives a call notice, the holder will not receive the coupon. However, once given the call notice, the holder will get the accrued interest if he elects to receive the call price. But, he will not receive the accrued interest if he chooses to convert into shares (this is known as the screw clause). Consequently, the value of the maximum constraint $V_{c,t}(S, t = t_c - T_n - \varepsilon)$ drops by the coupon value for $S \gg B_c$ (compared to $V_{c,t}(t = t_c - T_n + \varepsilon)$) and stays unchanged for $S \ll B_c$. The result of this drop backwards in time is a jump forwards in time. In some cases, $S^*(t) < B_c(t)$ (see Figure 5.2), but $V(t) \geq B_c$.

Thus, we can see that if a coupon payment is received within the notice period, S^* can be less than B_c. However, there are other cases where this effect can be observed. If $B_{cl} = F$, $S^* < B_c$ for large periods of time. Figure 5.6 shows a convertible with data in Table 5.1, but $B_{cl} = F = 100$, $T = 25$ and no dividends. The optimal call strategy is significantly lower than B_{cl} for the convertible without a default model. With the AFV default model, the optimal strategy S^* is higher, but near maturity, S^* also drops below the effective call price B_c. This result is consistent with Ingersoll's findings for perpetual convertible bonds with notice periods [Ing77b]. The approximation by Butler (Figure 5.6) is too high although one of the assumptions in the derivation of his result is that $B_{cl} = F$.

5.4 Implications on the value

It is interesting to examine the effect of the call policies on the CB value. In Figure 5.7, we can see the effect of different notice periods on the value of the convertible. These results are all obtained using an accurate PDE method (AFV model). The premium for a notice period varies over the stock price S with a maximum between 112 and 115. As predicted, the premium is larger for a longer notice period. The premium for a typical notice period with 30 days is about 0.90, a significant addition.

Another interesting subject is the effect of suboptimal call policies, especially the delayed call phenomena. Assuming that issuers call their convertibles late, what is the effect on the value. Consider the following strategy: The issuer calls only if it is beneficial for him to call. But, he will not call until the stock price level S_K is reached. This strategy is implemented by altering the model for valuation with notice periods. The maximum constraint in equation (3.4) respectively (3.16) becomes

$$\text{if } S \geq S_K \quad V \leq \max(V_{c,t}, \kappa S) \tag{5.11}$$
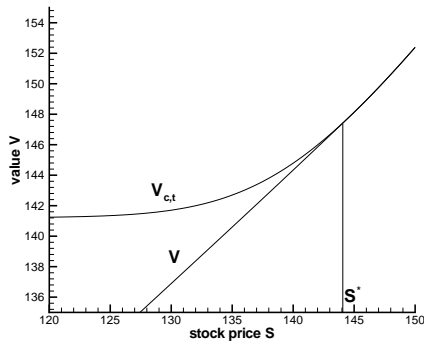
The impact of this new call strategy is presented in Figure 5.8. The difference in value compared with the optimal strategy for a convertible bond from Table 5.1 is shown over the stock price. This premium is very little for a strategy with $S_K = 150$ because the optimal strategy is close to this
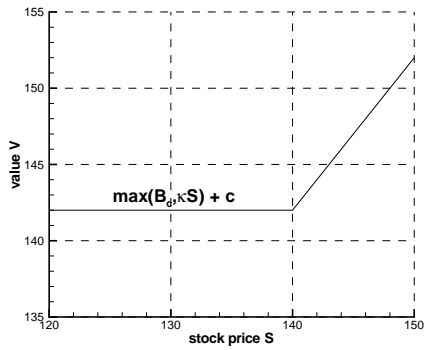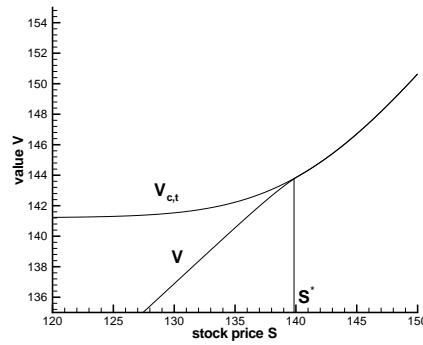
Initial condition: $V_{c,t}(\hat{t} = t_c + \varepsilon)$,
$t = t_c - T_n + \varepsilon$
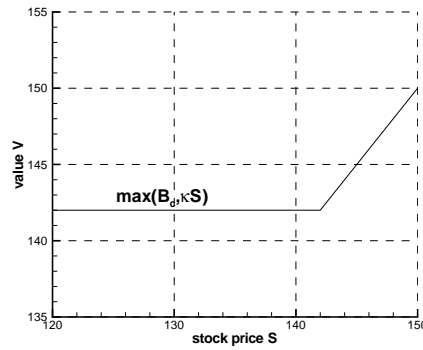
Initial condition: $V_{c,t}(\hat{t} = t_c - \varepsilon)$,
$t = t_c - T_n - \varepsilon$

Time $t = t_c - T_n + \varepsilon$

Time $t = t_c - T_n - \varepsilon$

Figure 5.5: *The price V and the constraint $V_{c,t}$ of a convertible with data is given in Table 5.1, but no dividends. The price is shown for $t = t_c - T_n - \varepsilon = 1.41666$ and $t = t_c - T_n + \varepsilon = 1.41668$, together with the initial conditions for the constraint $V_{c,t}$. The values are computed with the AFV model.*
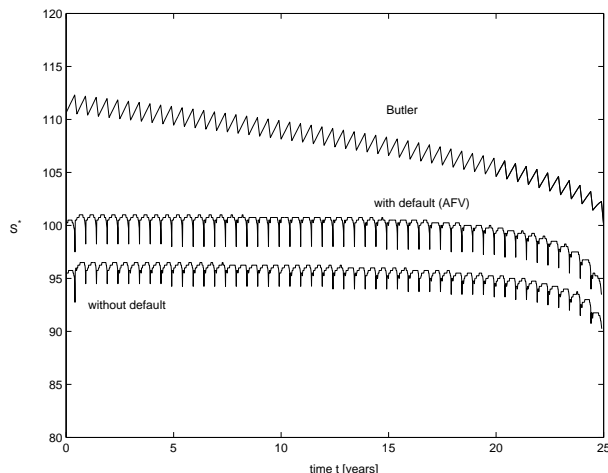
Figure 5.6: *The optimal call strategy S^* for a convertible with data in Table 5.1, but with maturity $T = 25$, $B_{cl} = 100$ and no dividends.*
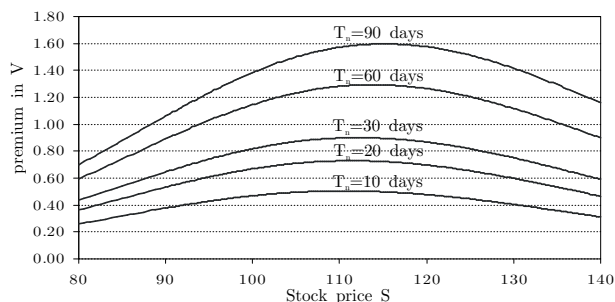

Figure 5.7: *The impact of different notice periods on the value: The difference in value compared to a convertible with $T_n = 0$ is shown over the stock price ($t = 0$). The AFV model with data in Table 5.1 is used.*

value (see Figure 5.4). But, for $S_K = 160$, we have a maximal impact on the value of more than 0.60. For $S_K = 170$, the value of the convertible increases by a maximum of about 0.60 compared with $S_K = 160$. That means that the optimality of the issuer's behavior has also a significant impact on the value of a convertible bond.

We now consider a more realistic scenario. Suppose the issuer uses an approximate method to determine the optimal call policy which we denote $\hat{S}$. This strategy can be modelled by replacing equation (3.5) (T&F model) or equation (3.16) (AFV model) by

$$\text{if } S \geq \hat{S}^*$$
$$V(S,t) = V_{c,t}(S,t). \tag{5.12}$$

Since this strategy is suboptimal, all values computed using equation (5.12) will be larger than values obtained with the optimal method. This makes the resulting premium a good measure of the error of approximating the optimal call strategy.
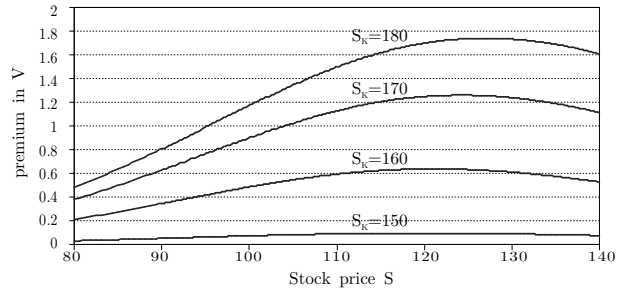
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Figure 5.8: *The impact of suboptimal call strategies (delay in S^*): The difference in value compared to an optimal called convertible is shown over the stock price ($t = 0$). The AFV model with data in Table 5.1 is used.*
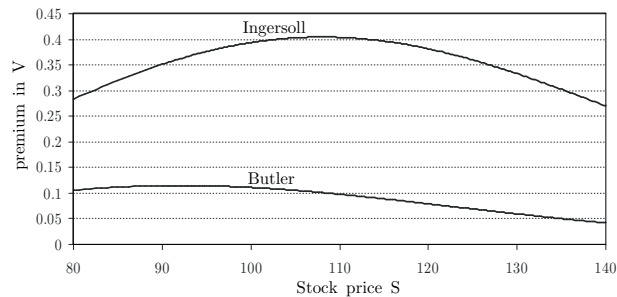

Figure 5.9: *The impact for suboptimal call strategies: Effect of Ingersoll's and Butler's policy on the value of a convertible is shown versus the stock price($t = 0$). An optimal call strategy has a premium of 0. The AFV model with data in Table 5.1 is used.*

Figure 5.9 shows the premium (compared to the optimal strategy) at $t = 0$ due to the different approximations. One can see that the call policy from Ingersoll has a significant effect on the value (about 0.40). The strategy from Butler has much smaller premium.

However, Butler's approximation depends critically on the call price B_c. In Figure 5.10 the call price is varied, and the stock price is held fixed at $S = 100$. We can see that Butler's approximation for S^* is poorer as B_c tends to the face value. This is surprising because one of the assumptions in the derivation of the approximation is $B_{cl} = F$.

6 Conclusions

Convertible bonds are a popular instrument with complex behavior. The notice period which prevents the issuer from an immediate call for conversion has a significant impact on the theoretical value of a convertible bond and the optimal call strategy of the issuer. A mathematical model that extends the existing model from Tsiveriotis and Fernandes (T&F model) and the model from Ayache, Forsyth and Vetzal (AFV model) is presented. The implementation leads to precise estimate for the value of the convertible and the optimal call strategy.

Various authors have analyzed the delayed call phenomena. They suggest that the issuers call
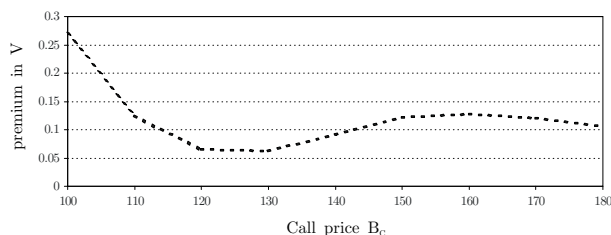
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Figure 5.10: *The premium in value of convertibles with different call prices using Butler's call strategy. An optimal call strategy has a premium of* 0. *The AFV model with data in Table 5.1 is used, and the call price varied.*

their convertibles above the optimal stock price level. If we assume such a delayed call, we find that the value of the convertible is larger than the convertible without a notice period. For example if the convertible is called 13% above the optimal value, with a notice period of 30 days, the value of the convertible increases by about 1% compared with the optimal strategy. A notice period of 30 days, assuming optimal issuer behavior, adds about 1% to the value compared to a bond with no notice period. Note that this effect is larger than the difference between T&F and AFV models.

Some authors argue that the introduction of a notice period results in a higher stock price level which is optimal for the issuer to call. We find that the call price and the schedule of coupon payments have a significant effect on this stock price level. In general, the optimal stock price is higher than the call price for convertibles with notice periods, but in some cases, it is lower. Especially just before a coupon payment is within the notice period, an optimal call by the issuer can be at a considerable lower stock price than the call price.

Various approximation methods for determining the optimal call policy have been discussed. Butler's method generally gives crude approximations to the actual value of the convertible. Only a full PDE method can provide accurate values, given the complex contractual details of typical convertible bonds.

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