# Decumulation of Defined Contribution Pension Plans: The Canasta Strategy

Peter A. Forsyth[a]

April 29, 2022

## 1 Introduction

Suppose an investor has diligently saved and invested in a defined contribution (DC) pension plan account. Upon retirement, the investor has to decide (i) on an investment strategy and (ii) a withdrawal schedule. However, the investor is now exposed to two significant sources of risk. It is likely that, in order to fund a reasonable lifestyle over the the retiree's lifespan (which could be 20-30 years), the investor will have some of her DC account invested in stocks. Coupled with the withdrawals required for expenses, this exposes the investor to a significant *sequence of return* risk (Milevsky and Posner, 2014). In addition, the retiree is also concerned about running out of savings before death, and hence is exposed to longevity risk. Consequently, Nobel laureate William Sharpe has referred to DC plan decumulation as *"the nastiest, hardest problem in finance"* (Ritholz, 2017).

The classic reference on DC plan decumulation is the work of Bengen (1994). Bengen (1994) showed that an investor who invested in a portfolio of 50% bonds and 50% stocks (rebalanced annually) could withdraw 4% of the initial account value (adjusted for inflation), and would have never run out of savings over any rolling 30 year historical period. This has given rise to the ubiquitous *four per cent rule* much beloved by financial advisors.

However, it is clear that instead of a constant proportion strategy (i.e. a fixed weight in stocks and bonds), an adaptive policy, whereby the stock-bond split responds to investment experience, can produce better results (Basu et al., 2011; Dang et al., 2017; Forsyth et al., 2019). Note that these papers assumed that the withdrawal amounts per year (in real terms) were fixed.

Another technique to optimize total withdrawals over the retiree's lifespan is to allow variable withdrawals (Waring and Siegel, 2015; Pfau, 2015; Tretiakova and Yamada, 2017; Forsyth et al., 2020). Finally, instead of some pre-determined variable withdrawal rule, one can treat the DC plan decumulation problem as an exercise in optimal stochastic control, with two controls (i) the amount withdrawn each year (subject to minimum and maximum constraints) and (ii) the stock-bond allocation (Forsyth, 2022).

[a]Cheriton School of Computer Science, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1, paforsyt@uwaterloo.ca

## 2 What is the point of this article?

A few years ago, I came across the obituary of Peter Ponzo, a Professor Emeritus from the University of Waterloo. Peter retired in 1993, and then took half his retirement savings and bought an annuity (at the time, annuities were priced at an interest rate of 9.8%). With the other half, he attempted to invest in stocks and beat the market. After all, he was a former Professor of Mathematics: how hard could this be?

Of course, Peter learned that investing was not so easy. He eventually devised a withdrawal strategy, which I have labelled the *canasta strategy*:

> *"If we have a good year, we take a trip to China, ..., if we have a bad year, we stay home and play canasta."* (retired professor Peter Ponzo, discussing his DC plan withdrawal strategy [1])

My objective here is to give you a mathematical formulation of the optimal stochastic control DC plan decumulation problem. The controls in this case are the asset allocation and withdrawal schedule. For some simple cases, we can in fact show that the canasta strategy is indeed optimal. I hope that you will find this interesting, and follow up by reading the complete paper (Forsyth, 2022).

This white paper contains some mathematics. If you have never heard of Ito's Lemma, and Partial Differential Equations (PDEs) aren't your thing, then perhaps the short blog on this topic might be more useful to you.[2]

## 3 Scenario

As an example, we consider a retiree with a stream of government benefits and DB plan payments. The retiree has a DC plan pension account, and requires a minimum withdrawal of $q_{\min}$ per year, which, coupled with these other annuity-like cash flows, provides for basic minimum expenses. The retiree also owns real estate, which can be regarded as a hedge against medical expenses (i.e. long term care), or as a hedge against left tail investment risk. To make this more concrete, we suppose that in the event that the DC plan is exhausted, and the retiree is still living, then cash can be realized by using a reverse mortgage on the real estate holdings. Mathematically, we will handle this case by allowing the DC plan account to become negative, and debt to accumulate at the borrowing rate.

In the event that a medical expense does not materialize, and there is no need to borrow due to poor investment returns, then the real estate is used as a bequest. In an act of mental accounting, the retiree does not regard the value of the real estate and his DC plan account as fungible assets.[3]

If the retiree's basic minimum expenses are covered, then the retiree may desire to withdraw at most $q_{\max}$ per year from the DC account. The amount that is withdrawn above $q_{\min}$ can be regarded as earmarked for discretionary spending (e.g. exotic vacations). We assume that the retiree is flexible about the precise timing of these discretionary withdrawals.

We focus solely on measured outcomes for the investment account, but as described above, it is easy to imagine that our retiree also owns real estate such as a home, so that investment and

[1] `https://www.theglobeandmail.com/report-on-business/math-prof-tests-investing-formulas-strategies/article22397218/`

[2] See `https://cs.uwaterloo.ca/~paforsyt/Canasta.html`

[3] My observation of several of my retired colleagues, as well as myself, verifies this act of mental accounting: real estate is considered to be separate from other investment assets.

longevity risk can be hedged using a reverse mortgage (using the home as collateral). However, we assume that the investor wants to avoid using a reverse mortgage if at all possible, so we seek an investment strategy that minimizes the risk of depleting the DC plan account. Our scenario shares some features with the behavioral life cycle approach originally described in Shefrin and Thaler (1988). In this framework, investors divide their wealth into separate "mental accounts" containing funds intended for different purposes such as current spending or future needs.

This is at odds with the standard life cycle approach which assumes that wealth is completely fungible across any such accounts, so that the same increase in wealth from any source (e.g. positive returns for a financial market portfolio, an increase in the value of one's house, lottery winnings, etc.) has the same effect on consumption.

In contrast, in the behavioral approach, wealth is not completely fungible, so the effects of increased wealth depend on the source of the increase. In our case, even if the investor's wealth rises because the value of his real estate has increased, there will be no impact on the amount withdrawn from the retirement portfolio. The real estate account will only be accessed as a last resort. It is assumed to be there in the background if needed, but it is ignored in our analysis.

# 4 Formulation

We assume that the investor has access to two funds: a broad market stock index fund and a constant maturity bond index fund.

The investment horizon is $T$. Let $S_t$ and $B_t$ respectively denote the real (inflation adjusted) *amounts* invested in the stock index and the bond index respectively. In general, these amounts will depend on the investor's strategy over time, as well as changes in the real unit prices of the assets. In the absence of an investor determined control (i.e. cash withdrawals or rebalancing), all changes in $S_t$ and $B_t$ result from changes in asset prices. We model the stock index as following a geometric Brownian motion (GBM).[4]

In the absence of control, $S_t$ evolves according to

$$\frac{dS_t}{S_t} \;\; = \;\; \mu^s \, dt + \sigma^s \, dZ^s \ , \tag{4.1}$$

where $\mu^s$ is the stock index drift rate, $\sigma^s$ is the volatility and $dZ^s$ is the increment of a Wiener process.

When I first looked at this topic, I assumed that the bonds were *risk-free assets* which had constant (real) returns. However, this was obviously unsatisfactory. I thought about this from the usual quantitative finance point view. First, I would need some sort of stochastic model of interest rates, and then use a bond pricing PDE to approximate the return of a constant maturity bond index. Them I would also need a stochastic model of inflation, and then convert the nominal return back to real. This looked difficult to calibrate.

Fortunately, one day I was chatting with one of our Waterloo PhD graduates, who has worked for several years in the financial industry. I explained to him my problem. He looked at me as if I was insane. *"Just assume that the returns of a constant maturity bond index follow geometric Brownian motion."* This is, of course, an eminently useful practitioner approach (Lin et al., 2015; MacMinn et al., 2014).

So, we directly model the (real) returns of the constant maturity bond index as a stochastic process. We assume that the constant maturity bond index follows GBM (this can be easily generalized).

[4]This is for ease of exposition. In Forsyth (2022), the stock index follows a jump diffusion.

Thus, $B_t$ evolves according to

$$\frac{dB_t}{B_t} = \left(\mu^b + \mu_c^b \mathbf{1}_{\{B_t<0\}}\right) dt + \sigma^b \, dZ^b, \tag{4.2}$$

where the terms in equation (4.2) are defined analogously to equation (4.1). The term $\mu_c^b \mathbf{1}_{\{B_{t^-}<0\}}$ in equation (4.2) represents the extra cost of borrowing (the spread).[5] The diffusion processes are correlated, i.e. $dZ^s \cdot dZ^b = \rho_{sb} \, dt$.

Let $W_t = S_t + B_t$ be the total wealth in the DC account. Assume that rebalancing is carried out continuously, with $p(W_t,t)$ being the fraction of wealth in the stock index

$$p(W_t, t) = \frac{S_t}{S_t + B_t} \ . \tag{4.3}$$

Assume continuous withdrawal of cash at a rate of $q(W_t, t)$ per year. The stochastic differential equation (SDE) for the total wealth process $W_t = S_t + B_t$ is then

$$dW_t = pW_t \frac{dS_t}{S_t} + (1-p)W_t \frac{dB_t}{B_t} - q \ dt \ . \tag{4.4}$$

# 5 Objective Function

The investor has two controls: the fraction of assets invested in stocks $p(W(t), t)$ and the withdrawal rate $q(W(t),t)$. Now, we need to define an objective function. As usual in financial applications, this will involve a measure of risk and reward.

We will assume that a 65-year old retiree makes the conservative assumption that she will live for 30 years. Labelling the current time $t = 0$ (when the retiree is 65), this means that the investment horizon is $[0,T]$ with $T = 30$ years (i.e. until age 95).

As a measure of reward, we will assume that the investor seeks to maximize the expected value of the total of the discounted withdrawals

$$EW = E\left[\int_0^T e^{-\rho t'} q(W(t'), t') \ dt'\right] \ , \tag{5.1}$$

where $E[\cdot]$ is the expectation. We include a discount rate $\rho$ in equation (5.1) for generality. We argue in Forsyth (2022) that, since all quantities are real (inflation adjusted) a conservative approach is $\rho = 0$.

For a measure of risk, we consider the value of the account (the total wealth) at time $T$, which we denote by $W_T$. Since the retiree's main concern is running out of cash (and hence having to borrow against real estate holdings) we can use various risk measures, such as expected shortfall (i.e. the mean of the worst $\alpha$ fraction outcomes), probability of ruin, or quadratic shortfall. We denote this risk measure by $ER$. We can determine $ER$ by computing

$$ER = E\left[e^{-\rho T} \mathcal{R}(W_T)\right] \tag{5.2}$$

where the function $\mathcal{R}(W_T)$ depends on the precise risk measure specified. We have also included a discounting term here to be consistent with equation (5.1).

[5] Recall that the DC plan account can become negative, due to the minimum withdrawal required. A negative DC account balance is assumed to funded by a reverse mortgage on the retiree's real estate holdings.

We are deliberately vague about the precise risk measure used. Our main objective in this article is to determine when the *canasta* strategy is optimal. Rather counter-intuitively, it will turn out that optimality of the canasta strategy depends only on the precise form of $EW$, but not $ER$.

As an example, suppose we used probability of ruin as a risk measure, then

$$\mathcal{R}(W_T) \quad = \quad \mathbf{1}_{\{W_T < 0\}} \ , \tag{5.3}$$

so that

$$E\Big[e^{-\rho T}\mathcal{R}(W_T)\Big] \quad = \quad e^{-\rho T} Prob[W_T < 0] \tag{5.4}$$

But, probability of ruin does not weight the losses. Recall that negative account values correspond to borrowing against real estate. Having to borrow \$10,000 is worse than having to borrow \$10. Perhaps a better idea would be

$$\mathcal{R}(W_T) \quad = \quad \big|W_T\big|\mathbf{1}_{\{W_T < 0\}} \ , \tag{5.5}$$

so that

$$E\Big[e^{-\rho T}\mathcal{R}(W_T)\Big] \quad = \quad e^{-\rho T} E\Big[\big|\min(W_T, 0)\big|\Big] \ . \tag{5.6}$$

Note that the absolute value in (5.5) means that larger losses are weighted larger, i.e. a large value of $|\min(W_T{,}0)|$ is undesirable. Another idea might be to use an expected shortfall type risk measure, i.e. the mean of the worst $\alpha$ fraction of outcomes. But, you get the idea. This will all be encapsulated in some function $\mathcal{R}(W_T)$.

Since expected withdrawals $EW$ and expected risk $ER$ are conflicting measures, we use a scalarization technique to find the Pareto points for this multi-objective optimization problem. Informally, for a given parameter $\kappa > 0$, we seek the controls $(p(W(t){,}t), q(W(t){,}t))$ that maximize

$$\text{Objective Function} \quad = \quad EW - \ \kappa ER. \tag{5.7}$$

Varying $\kappa$ will trace out an efficient frontier in the $(EW, ER)$ plane.

# 6 Optimal Control

We define the set of possible values for the control $q$ (withdrawal rate)

$$q \in \mathcal{Z}_q \quad = \quad \begin{cases} [q_{\min}, q_{\max}] & W_t > 0 \\ q_{\min} & W_t \leq 0 \end{cases} \ . \tag{6.1}$$

Note that if the DC plan account is depleted ($W_t \leq 0$), we will continue to withdraw from the DC account (at the minimum rate). Recall that in this case, this is a virtual withdrawal, since we actually borrow using the (assumed) family home as collateral. In effect, the negative amount in the DC account accumulates as debt, financed by the borrowing rate.

The set of possible values for the fraction in stocks are

$$p \in \mathcal{Z}_p(W_t{,}t) \quad = \quad \begin{cases} [0{,}1] & W_t > 0 \ ; \ t \in [0{,}T] \\ \{0\} & W_t \leq 0 \ ; \ t \in [0{,}T] \end{cases} \ , \tag{6.2}$$

where we have specified that $p \in [0,1]$, i.e. no shorting, no leverage. We also require all stocks to be sold if the DC account becomes negative due to the required minimum withdrawals.

We will determine the optimal controls by dynamic programming. To this end, we define the value function $V(w,t)$ on the domain $\Omega = (-\infty, +\infty) \times [0,T]$ as

$$V(w,t) = \sup_{p(\cdot)\in\mathcal{Z}_p} \sup_{q(\cdot)\in\mathcal{Z}_q} \left\{ E_{(p,q)}^{(W_t,t)} \left[ \int_t^T e^{-\rho(t'-t)} q(W(t'),t')\, dt' - \kappa\, e^{-\rho(T-t)} \mathcal{R}(W_T) \middle| W_t = w \right] \right\} . \tag{6.3}$$

where $\mathcal{R}(W_t)$ is a measure of risk.

Under these assumptions, we can derive the following result

**Proposition 6.1** (Bang-bang control for withdrawals)**.** *Assume that*

- *the stock and bond processes follow (4.1) and (4.2) ;*
- *the portfolio is continuously rebalanced and withdrawals occur at a continuous (finite) rate* $q \in [q_{\max}, q_{\min}]$*;*
- *in the event of ties, the control* $q_{\min}$ *is selected,*

*then the optimal control for problem (6.3) is bang-bang, i.e. the optimal choice for* $q$ *is either* $q = q_{\min}$ *or* $q = q_{\max}$*, independent of any choice of risk measure* $\mathcal{R}(W_T)$*.*

*Proof.* See Appendix A □

So we can see that Professor Ponzo was correct, at least for this simple case. The retiree should only choose to withdraw at the maximum or minimum rates, nothing in between. Basically, if stocks do well, you should withdraw at the maximum rate (and take exotic vacations) to crystallize these gains. Otherwise, you should withdraw at the minimum rate, to ameliorate sequence of return risk (Milevsky and Posner, 2014).

# 7 Generalization of Proposition 6.1

In Forsyth (2022), Proposition 6.1 was generalized by assuming jump diffusion processes for the stock and bond indexes. In fact, you can assume stochastic volatility processes, Hawkes processes, or anything reasonable, and the result still holds.

More realistically, investors withdraw fixed amounts at monthly or yearly intervals (not continuously) and rebalance their portfolios at discrete intervals as well. In this case, we can't obtain closed form results for the withdrawal controls. However, if we solve the optimal control problem numerically (without assuming bang-bang controls) then, it seems that, for practical purposes, the withdrawal controls are still bang-bang. Figure 7.1 shows the optimal withdrawal heat map for an example assuming discrete withdrawals and rebalancing intervals, from (Forsyth, 2022). Note the very small transition zone between withdrawing at $q_{\min}$ and $q_{\max}$.[6]

[6] To be precise here, in this case, $q_{\min}$ and $q_{\max}$ are withdrawal *amounts* not rates.

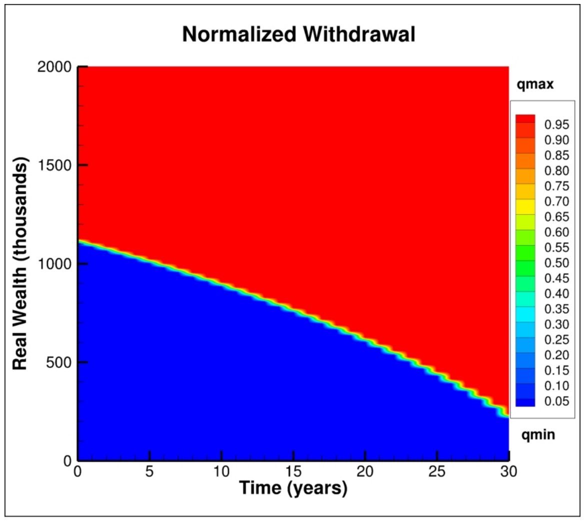


FIGURE 7.1: *Heat map of optimal withdrawal controls from (Forsyth, 2022). Discrete rebalancing and discrete withdrawals. The numerical method did not assume bang-bang controls.*

# 8 Why are the $q$ controls bang-bang?

Suppose we replaced the reward equation (5.1) by

$$EW = E\left[\int_0^T e^{-\rho t'} \mathcal{U}\big(q(W(t'),t')\big)\ dt'\right] , \tag{8.1}$$

where $\mathcal{U}(q)$ is a nonlinear utility function. Recalling that $V(w,t)$ is the value function in equation (6.3), and following the argument in Appendix A, we find that the optimal withdrawal control $q$ is determined from[7]

$$\sup_{q \in \mathcal{Z}_q} \left(\mathcal{U}(q) - qV_w\right) . \tag{8.2}$$

so that, in general, the optimal value of $q$ would not be either $q_{\min}$ or $q_{\max}$, i.e. not bang-bang. So, the bang-bang control optimality depends crucially on the assumption that the reward is a linear function of the withdrawals.

Is this realistic? First of all, recall that all quantities are real, so we are postulating that the reward is in terms of real future dollars. Speaking as a retiree myself, I know that when I plan an exotic vacation, I need to pay in real dollars. I have not encountered a cruise line which accepts *utils* as payment.[8] It therefore seems eminently reasonable to base the reward on a linear function of the withdrawal amounts.[9]

Suppose $\mathcal{U}(q)$ is a piecewise linear function of $q$. Then, the optimal values of $q$ occur at the breakpoints of the piecewise linear function (as well as the endpoints $q_{\min}, q_{\max}$). So, again, the control is bang-bang (i.e. the optimal values of $q$ are drawn from a finite set).

[7]Note that at any instant in time, we can regard $V_w$ as fixed.

[8]My spouse, who is an economist, rolls her eyes when I mention this. I contend that utility functions are simply a naive mathematical convenience for economists, which fail to take into account actual human behavior. We haven't filed for divorce yet, but the dinner time conversations sometimes become testy.

[9]My thinking is that the idea of a utility function was to embed, in a single simple function, the tradeoff between risk and reward. This, of course, facilitates closed form solutions, much beloved by academics. However, we have precisely split apart these two measures, in terms of EW and ER. This allows us, in particular, to specify the type of risk we are minimizing. In the decumulation problem, most retirees are concerned with running out of cash, so we focus on these types of risk measures, which, we contend, are not well modelled by the common utility functions.

Let's summarize this result

**Proposition 8.1** (Bang-bang control for piecewise linear utility functions)**.** *Assume that*

- *the stock and bond processes follow (4.1) and (4.2) ;*
- *the portfolio is continuously rebalanced and withdrawals occur at a continuous (finite) rate* $q \in [q_{\max}, q_{\min}]$*;*
- *in the event of ties, the smallest control is selected;*
- *the reward is measured in terms of a piecewise linear utility function* $\mathcal{U}(q)$*,*

*then the optimal control for problem (6.3) is bang-bang, i.e. the optimal choice for* $q$ *is either* $q = q_{\min}$ *or* $q = q_{\max}$*, or at the breakpoints of the utility function, independent of any choice of risk measure* $\mathcal{R}(W_T)$*.*

*Proof.* This follows from equation (8.2). □

# 9 What about taxes?

Suppose that we withdraw annually, at times $t_i, t_i + \Delta t, ...$ where $\Delta t = 1$ year. In this case, the reward function would be

$$\begin{aligned} EW &= E\left[\sum_i e^{-\rho t_i} \hat{q}(W(t_i), t_i)\right] , \\ & \hat{q}_i \in [\hat{q}_{\min}, \hat{q}_{\max}] \end{aligned} \tag{9.1}$$

where $\hat{q}$ is now a dollar amount of withdrawal, not a rate of withdrawal. To be precise here, let $W_i^+$ be the total wealth in the DC account after withdrawals, and $W_i^-$ be the total wealth just before withdrawals (at $t_i$), then we have

$$W_i^+ = W_i^- - \hat{q}_i \ . \tag{9.2}$$

In Canada, a tax advantaged account (an RRSP) defers taxes until cash is withdrawn, which is then taxed as ordinary income. Suppose we are concerned with maximizing the total *after tax* withdrawals. Let $\mathcal{A}(q)$ be the after tax income after withdrawing $q$ dollars annually (this function is easy to construct, using the marginal tax tables). Then, assuming annual withdrawals, the reward function would become

$$\begin{aligned} EW &= E\left[\sum_i e^{-\rho t_i} \mathcal{A}(\hat{q}_i)\right] , \\ & \hat{q}_i \begin{cases} \in [\hat{q}_{\min}, \hat{q}_{\max}] & W_t > 0 \\ = \hat{q}_{\min} & W_t \leq 0 \end{cases} \end{aligned} \tag{9.3}$$

where we minimize the amount of borrowing when $W_t \leq 0$.[10] However, because we are borrowing, we don't need to borrow $\hat{q}_{\min}$ to receive $\mathcal{A}(\hat{q}_{\min})$ dollars, we need to borrow only $\mathcal{A}(\hat{q}_{\min})$ (since we don't pay tax on borrowed dollars). So, we adjust equation (9.2) to be

$$W_i^+ = \begin{cases} W_i^- - \hat{q}_i & W_t > 0 \\ W_i^- - \mathcal{A}(\hat{q}_{\min}) & W_t \leq 0 \end{cases} \tag{9.4}$$

[10] Recall that we assume that the investor borrows against her real estate if $W_t \leq 0$. In this case ($W_t \leq 0$), we assume that the retiree borrows $\hat{q}_{\min}$ (the minimum necessary) to cover expenses.

In this case, $\hat{q}_i \in [\hat{q}_{\min}, \hat{q}_{\max}]$ is a virtual control, with the actual cash flows given by equations (9.3-9.4).

In Canada, if you pass away with money still left in your DC account, then your final tax return has a deemed withdrawal of all remaining cash in your account, which then appears as income on your final tax return in the year of death. To take this into account, then probably we should have

$$\overbrace{\mathcal{R}(W_T)}^{after\ tax} = \begin{cases} \mathcal{R}\big(\mathcal{A}(W_T)\big) & W_T > 0 \\ \mathcal{R}(W_T) & W_T \leq 0 \end{cases} \tag{9.5}$$

where we note that there are no taxes applied to debt. So, this is the after tax value of the remaining cash to your heirs, taking into account any possible liability against the real estate collateral.

Now, from Proposition 8.1, we know that in the continuous withdrawal limit, the optimal withdrawal strategy for a piecewise linear utility function is to withdraw at $q_{\min}, q_{\max}$ or the breakpoints of the piecewise linear utility function.

The after-tax function, in the discrete withdrawal case, is piecewise linear.[11] So, we would expect that the optimal withdrawal control (assuming the objective function is based on after tax withdrawals) is also (approximately) bang-bang in this case as well. But this remains to be seen.

Bottom line: taxes introduce some interesting optimal control problems. This needs more attention from the academic community.

## 10 Conclusion

Mathematically, we can prove (in simple cases) that the optimal DC plan withdrawal controls are *bang-bang*, i.e. withdraw only at the minimum rate or the maximum rate, pretty much independent of whatever risk measure you use. This is fundamentally due to the reward measure which is linear in the (real) withdrawal amounts. If we consider more realistic stochastic models for the stock and bond processes, as well as discrete rebalancing, then numerical computations show that, for practical purposes, the optimal withdrawals are still bang-bang.

If we consider after-tax dollars, then, since the after-tax *utility function* is piecewise linear, it is likely that the discrete withdrawal controls are still approximately bang-bang. But this remains to be verified.

Professor Ponzo's essential idea: withdraw the maximum from your retirement portfolio when stocks do well, and withdraw the minimum when stocks do poorly, is basically optimal. The idea is to take money off the table (and have a good time) if you are lucky (i.e. convert your paper gains to consumption). Otherwise, sit tight and withdraw the minimum amount required to pay expenses.

# Appendix

## A Proof of Proposition 6.1

*Proof.* Consider the problem posed in equation (6.3). From the tower property of expectations we have (for $h \to 0$)

[11] This is certainly true in Canada. This is because the graduated income tax is specified by a list of tax brackets, with a constant marginal tax rate in each bracket. I suspect that most income tax specifications in many countries follow these sorts of rules.

$$V(w,t) = \sup_{p(\cdot)\in\mathcal{Z}_p} \sup_{q(\cdot)\in\mathcal{Z}_q} \left\{ E_{(p,q)}^{(W_t,t)} \left[ e^{-\rho h} \Big( V\big(W(t+h), t+h\big) + hq(t+h) \Big) \Big| W(t) = w \right] \right\} \quad \text{(A.1)}$$

Noting the SDEs (4.1-4.2), and the final equation (4.4), then applying Ito's Lemma to equation (A.1), and letting $h \to 0$, gives the Hamilton-Jacobi-Bellman PDE for $V$

$$\begin{aligned} V_t + \sup_{p\in\mathcal{Z}_p} \sup_{q\in\mathcal{Z}_q} \Bigg\{ w \left[ p\mu^s + (1-p)(\mu^b + \mu_c^b \mathbf{1}_{\{w<0\}}) \right] V_w - qV_w + q \\ + w^2 \left[ \frac{(p\sigma^s)^2}{2} + (1-p)p\sigma^s\sigma^b\rho_{sb} + \frac{((1-p)\sigma^b)^2}{2} \right] V_{ww} - \rho V \Bigg\} = 0 \ , \end{aligned} \quad \text{(A.2)}$$

with terminal condition

$$V(w,T) = -\kappa \mathcal{R}(w,T) \ . \quad \text{(A.3)}$$

In general, we seek the viscosity solution(Crandall et al., 1992; Barles and Souganidis, 1991; Barles et al., 1995) of equation (A.2), which does not require that the solution $V$ be differentiable. However, we make the assumption that $V_w$ exists and is bounded.

Rewriting equation (A.2) we have

$$\begin{aligned} V_t + \sup_{p\in\mathcal{Z}_p} \Bigg\{ w \left[ p\mu^s + (1-p)(\mu^b + \mu_c^b \mathbf{1}_{\{w<0\}}) \right] V_w + w^2 \left[ \frac{(p\sigma^s)^2}{2} + (1-p)p\sigma^s\sigma^b\rho_{sb} + \frac{((1-p)\sigma^b)^2}{2} \right] V_{ww} \Bigg\} \\ + \sup_{q\in\mathcal{Z}_q} \Big\{ q(1-V_w) \Big\} - \rho V = 0 \ , \end{aligned} \quad \text{(A.4)}$$

and therefore the optimal value of $q$ is determined by maximizing

$$\sup_{q\in\mathcal{Z}_q} q(1-V_w) \ . \quad \text{(A.5)}$$

Breaking ties by choosing $q = q_{\min}$ if $(1-V_w) = 0$.[12], we then have that the optimal strategy $q^*$ is

$$q^* = \begin{cases} q_{\min} \ ; & (1-V_w) \le 0 \ , w > 0 \\ q_{\max} \ ; & (1-V_w) > 0 \ , w > 0 \\ q_{\min} \ ; & w \le 0 \end{cases} \ . \quad \text{(A.6)}$$

□

# References


Barles, G., C. Daher, and M. Romano (1995). Convergence of numerical schemes for parabolic equations arising in finance theory. *Mathematical models and methods in applied Sciences 5*(01), 125–143.


[12]We also require that $q = q_{\min}$ if $w \le 0$ to minimize borrowing

Barles, G. and P. E. Souganidis (1991). Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic analysis 4*(3), 271–283.

Basu, A. K., A. Byrne, and M. E. Drew (2011). Dynamic lifecycle strategies for target date retirement funds. *Journal of Portfolio Management 37*(2), 83–96.

Bengen, W. (1994). Determining withdrawal rates using historical data. *Journal of Financial Planning 7*, 171–180.

Crandall, M. G., H. Ishii, and P.-L. Lions (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society 27*, 1–67.

Dang, D.-M., P. A. Forsyth, and K. R. Vetzal (2017). The 4% strategy revisited: a pre-commitment optimal mean-variance approach to wealth management. *Quantitative Finance 17*, 335–351.

Forsyth, P. A. (2022). A stochastic control approach to defined contribution plan decumulation: "The Nastiest, Hardest Problem in Finance". *North American Actuarial Journal*. to appear.

Forsyth, P. A., K. R. Vetzal, and G. Westmacott (2019). Management of portfolio depletion risk through optimal life cycle asset allocation. *North American Actuarial Journal 23:3*, 447–468.

Forsyth, P. A., K. R. Vetzal, and G. Westmacott (2020). Optimal asset allocation for a DC pension decumulation with a variable spending rule. *ASTIN Bulletin 50:2*, 419–447.

Lin, Y., R. MacMinn, and R. Tian (2015). De-risking defined benefit plans. *Insurance: Mathematics and Economics 63*, 52–65.

MacMinn, R., P. Brockett, J. Wang, Y. Lin, and R. Tian (2014). The securitization of longevity risk and its implications for retirement security. In O. S. Mitchell, R. Maurer, and P. B. Hammond (Eds.), *Recreating Sustainable Retirement*, pp. 134–160. Oxford: Oxford University Press.

Milevsky, M. and S. Posner (2014). Can collars reduce retirement sequencing risk? *Journal of Retirement 1(4)*, 46–56.

Pfau, W. D. (2015). Making sense out of variable spending strategies for retirees. *Journal of Financial Planning 28*(10), 42–51.

Ritholz, B. (2017). Tackling the 'nastiest, hardest problem in finance'. `www.bloomberg.com/view/articles/2017-06-05/tackling-the-nastiest-hardest-problem-in-finance`.

Shefrin, H. M. and R. H. Thaler (1988). The behavioral life-cycle hypothesis. *Economic Inquiry 26*, 609–643.

Tretiakova, I. and M. S. Yamada (2017). Autonomous portfolio: A decumulation investment strategy that will get you there. *Journal of Retirement 5:2*, 83–95.

Waring, M. B. and L. B. Siegel (2015). The only spending rule article you will ever need. *Financial Analysts Journal 71*(1), 91–107.