A Buy and Hold Portfolio Loses Diversification

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Abstract

Two recent papers describe some intriguing empirical results (Bessembinder (2018) JFE) and (Farago and Hjalmarsson (2023) RAPS). Basically, the majority of stocks perform poorly compared to T-bills. Most of the stock market gains can be attributed to a small number of stocks. Even randomly selected, small portfolios (i.e. 50 stocks), equally weighted, can outperform the market portfolio. Simulations in a simplified theoretical market are used to explain some of these results. In this paper, we examine the theoretical market suggested in (Farago and Hjalmarsson (2023) RAPS) in more detail. We consider a model market with 100 stocks, each following Geometric Brownian Motion (GBM). All the stocks have the same expected return, volatility and pairwise correlation. At time zero, an equal amount is invested in each stock, with no further trading. This buy and hold portfolio corresponds to the market portfolio in this case. After 30 years, 95% of the final portfolio value is concentrated in just 16 stocks (out of 100 stocks). Only 20 stocks have positive Internal Rate of Return (IRR). An equal weight strategy partially stochastically dominates the buy and hold portfolio.

Keywords: Skewed compound returns, stock market concentration, equal weighting, volatility pumping

1 Introduction

This white paper is motivated by several recent publications. Bessembinder (2018) notes that empirical analysis suggests that most individual stocks are losers during their lifetime. Over the period 1926-2016, Bessembinder (2018) shows that most of the wealth creation in the stock market can be attributed to a small number of firms. Some interesting facts from Bessembinder (2018):

- individual stocks tend to have short lifetimes. The median time that a stock is listed on the Center for Security Prices (CRSP) database (1926-2016) is less than eight years;
- over their lifetimes, less than 43% of stocks (including reinvested dividends) outperform T-bills;
- the CRSP database holds 23,500 stocks (all stocks which traded during 1926-2016). Of these stocks, just 4% (the top lifetime performers) accounted for all the net long-term wealth creation in the stock market (i.e. accumulated value in excess of investing in T-bills). The remaining 96% of stocks collectively matched T-bill returns (some stocks exceeded T-bills, many were wealth destroyers).

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Following along this theme, Farago and Hjalmarsson (2023) point out that since the return distribution of a buy and hold portfolio is highly skewed (due to the approximate log-normality of cumulative returns), small, equally weighted portfolios can be superior to the buy and hold portfolio (i.e. the capitalization weighted index) with high probability. Perhaps the most surprising result is the following:

- the CRSP data for 1979-2016 is used to generate blocks of 30 year returns. A portfolio picking 50 stocks at random, and rebalancing to equal weighting each month, beats the market (the capitalization weighted CRSP index) 96% of the time over 30 years.

Farago and Hjalmarsson (2023) emphasize that this non-intuitive result is largely due to the equal weight rebalancing effect. They use simulations based on simple assumptions to explain this result.

Finally, we note the results in Forsyth (2022). Here, block bootstrap simulations (Politis and Romano, 1994; Politis and White, 2004; Dichtl et al., 2016; Anarkulova et al., 2022) are used based on the CRSP data over 1926-2022,[1] to generate the distribution of

(i) a portfolio invested in 60% CRSP *capitalization* weighted index, and 40% T-bills, annually rebalanced;

(ii) a portfolio invested in 60% *equally weighted* CRSP index, and 40% T-bills, annually rebalanced.

All returns were deflated by the CPI, so represent real returns. A 30-year investment horizon was considered.

Using the full 1926-2022 CRSP data, Figure 1.1(a) shows that portfolio (ii) stochastically dominates (to first order) portfolio (i). In other words, independent of any individual investor utility function, any investor who prefers more rather than less, will prefer portfolio (ii) to portfolio (i).[2]

However, if we restrict attention to CRSP data from 1980-2022, and again look at 30 year bootstrapped investment horizons, we obtain the results in Figure 1.1(b) which shows the stochastic dominance of portfolio (ii) over (i) has almost disappeared. However, using only 40 years of data as the basis for bootstrapping 30 year returns is a bit dubious. Nevertheless, this is consistent with recent market behaviour, with market capitalization becoming increasingly concentrated. For example, recent S&P 500 gains can be attributed entirely to the *magnificent seven* stocks.[3]

The long term superior returns generated by equal weight portfolios is discussed in DiMeguel et al. (2009); Tljaard and Mare (2021); Plyakha et al. (2021); Forsyth (2022). Plyakha et al. (2021) note that a large part of the enhanced return of the equal weight strategy is due to the rebalancing, not just the small-cap effect.

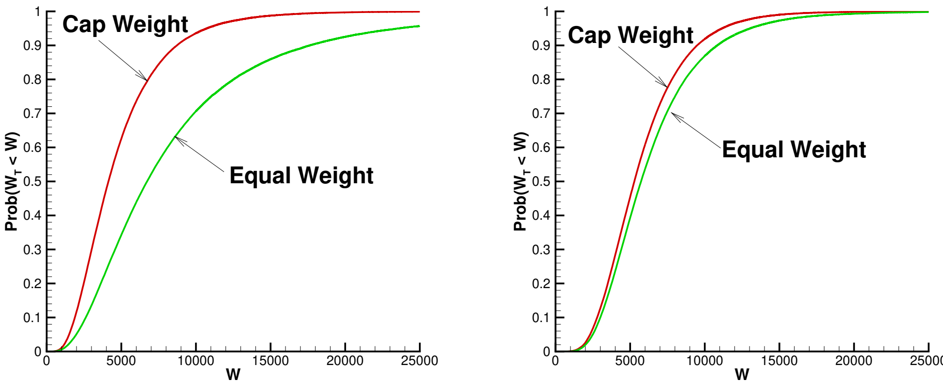
1.1 Our objective in this paper

Many of the papers described above seem to indicate that portfolios which rebalance to equal weights is a good idea. Bessembinder (2018) and Farago and Hjalmarsson (2023) suggest that many of their unexpected results can be explained by the skewness of long-term stock returns (i.e. a small number of hugely out-performing stocks, along with a large number of mediocre stocks). Farago

[1]More specifically, results presented here were calculated based on data from Historical Indexes, ©2022 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services (WRDS) was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

[2]This is a good example of the non-usefulness of Sharpe ratios for compound long-term returns. Portfolio (ii) dominates portfolio (i), but portfolio (i) has a larger Sharpe ratio compared to portfolio (ii).

[3]At pixel time, Apple, Amazon, Alphabet, Meta, Microsoft, Tesla, and Nvida make up about 30% of the S&P 500 by market capitalization.



(a) *CRSP data: 1926:1-2022:1* (b) *CRSP data: 1980:1-2022:1*

FIGURE 1.1: *Cumulative distribution functions (CDFs) for cap weighted and equal weighted indexes, as a function of final real wealth W at $T = 30$ years. Initial stake $W_0 = 1000$, no cash injections or withdrawals. Block bootstrap resampling, expected blocksize 2.0 years. 60% stocks, 40% bonds, rebalanced annually. Bond index: 30 day US T-bills. Stock index: CRSP capitalization weighted or CRSP equal weighted index. Data range shown. All indexes are deflated by the CPI. 10^6 resamples. Range of CRSP data shown.*

and Hjalmarsson (2023) attempt to explain this using an example of a portfolio of stocks following geometric Brownian motion (GBM).[4]

Based on the maxim " *All models are wrong, some are useful,"* we will examine this model in some detail in this paper.

2 A simplified theoretical market

Suppose we have a market with N assets $S_i; i = 1,..,N$ which follow

$$\begin{aligned} dS_i &= \mu_i S_i \; dt + \sigma S_i \; dZ_i \\ & \quad dZ_i \cdot dZ_j = \rho_{ij} \; dt \; , \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} S_i &= \text{price of asset } i \\ \sigma_i &= \text{volatility of asset } i \\ \mu_i &= \text{arithmetic return of asset } i \\ dZ_i &= \text{increment of a Wiener process} \\ \rho_{ij} &= \text{correlation between assets } i,j \; . \end{aligned} \tag{2.2}$$

We assume that no stocks enter or exit this market, no dividends are paid, and there are no cash injections/withdrawals after investing the initial capital.

[4]Note that the results in Bessembinder (2018) and Farago and Hjalmarsson (2023) are backed up by empirical tests.

2.1 Buy and hold

At $t = 0$, we purchase n_i units of each asset i, and simply buy and hold these assets. We will evaluate our total wealth at time T. Our initial wealth will be

$$W(0) = \sum_{i=1}^{n} n_i S_i(0) \ . \tag{2.3}$$

At time zero, we assume

$$\begin{aligned} S_i(0) &= 1.0 \ ; \ i = 1, \ldots, N \\ n_i &= 1/N \ ; \ i = 1, \ldots, N \ , \end{aligned} \tag{2.4}$$

which implies that

$$W(0) = 1.0 \ . \tag{2.5}$$

So, initially, we allocate cash to all assets equally. Of course, our allocation to each asset will change over time, as the assets evolve according to equation (2.1) and will no longer be equal weighted at $t = T$. This buy and hold strategy in this case corresponds to holding the capitalization weighted market index.

At time T we have (using equation (2.4))

$$W(T) = \frac{1}{N} \sum_{i=1}^{N} S_i(T) \ . \tag{2.6}$$

For each asset S_i, it follows from equation (2.1) that the final asset values $S_i(T)$ are log-normally distributed (noting equation (2.4))

$$\begin{aligned} S_i(T) &= S_i(0) \exp\Big((\mu_i - \sigma_i^2/2)T + \sigma_i (Z_i(T) - Z_i(0)) \Big) \\ &= \exp\Big((\mu_i - \sigma_i^2/2)T + \sigma_i (Z_i(T) - Z_i(0)) \Big) \ , \end{aligned} \tag{2.7}$$

where $E[\cdot]$ is the expectation operator, and

$$Z_i(T) - Z_i(0) \simeq \mathcal{N}(0,T) \ . \tag{2.8}$$

$\mathcal{N}(0,T)$ is a draw from a normal distribution with mean zero and variance T.

It is straightforward to show that

$$\begin{aligned} E[S_i(T)] &= \exp(\mu_i T) \\ Median[S_i(T)] &= \exp\Big((\mu_i - \sigma_i^2/2)T \Big) \\ std[S_i(T)] &= \exp(\mu_i T) \sqrt{(e^{\sigma_i^2 T} - 1)} \ , \end{aligned} \tag{2.9}$$

and therefore that

$$\begin{aligned} E[W_T] &= \frac{1}{N} E[\sum_i S_i(T)] \\ &= \frac{1}{N} \sum_{i=1}^{N} \exp(\mu_i T) \ , \end{aligned} \tag{2.10}$$

and where we have used equation (2.6).

Note that the sum of log-normals is not log-normal (in general), so we cannot determine any other properties of W_T in closed form. We will have to resort to simulation in order obtain other statistics.

2.2 Rebalance to equal weight

Now, we consider the case where we rebalance our investment portfolio back to the original equal capitalization weight, at each instant in time.[5] Suppose our investment strategy at time t is to invest $\hat{n}_i(t)$ in each asset, where now $\hat{n}_i(t, W(t), S_i(t))$ is a function of (t, W, S) in general. Let

$$W(t) = \sum_i \hat{n}_i(t) S_i(t) \tag{2.11}$$

be the value of our portfolio at t. Let $G = \log(W(t))$. Using Ito's Lemma and equation (2.1), gives

$$dG = \left[\sum_k \frac{\hat{n}_k \mu_k S_k}{W} - \frac{1}{2} \sum_{k,m} \frac{\hat{n}_k \hat{n}_m S_k S_m \sigma_k \sigma_m \rho_{km}}{W^2} \right] dt + \sum_k \frac{\hat{n}_k S_k \sigma_k}{W} \, dZ_k \ . \tag{2.12}$$

Now, suppose we choose a constant proportions strategy, i.e we rebalance at every instant in time so that we have a constant weight w_i in each asset,

$$\begin{aligned} w_i &= \frac{\hat{n}_i S_i}{W(t)} \\ &\sum_i w_i = 1 \ . \end{aligned} \tag{2.13}$$

Note that w_i is independent of t, since we rebalance at every opportunity. We can then write equation (2.12) as (using equation (2.13))

$$dG = \left[\sum_k w_k \mu_k - \frac{1}{2} \sum_{k,m} w_k w_m \sigma_k \sigma_m \rho_{km} \right] dt + \sum_k w_k \sigma_k \, dZ_k \ . \tag{2.14}$$

Equation (2.14) has the exact solution

$$\begin{aligned} G(t) &= G(0) + \\ &\left[\sum_k w_k \mu_k - \frac{1}{2} \sum_{k,m} w_k w_m \sigma_k \sigma_m \rho_{km} \right] t + \sum_k w_k \sigma_k (Z_k(t) - Z_k(0)) \ . \end{aligned} \tag{2.15}$$

We now assume that we use an equal weight strategy, i.e. we rebalance so that we always allocate the same amount of wealth to each asset

$$w_i = \frac{1}{N} \ ; \ i = 1, \ldots, N \ . \tag{2.16}$$

Initially, we allocate equal amounts of cash to each asset (the same assumption as used for the buy and hold case),

$$\begin{aligned} S_i(0) &= 1.0 \\ \hat{n}_i(0) &= \tfrac{1}{N} \\ W(0) &= 1.0 \ . \end{aligned} \tag{2.17}$$

[5] Continuous rebalancing is assumed for mathematical convenience. However, Farago and Hjalmarsson (2023) show that changing their rebalancing period from one month to one year, for a 30 year investment horizon, does not change the simulation results appreciably.

Using equation (2.16) in equation (2.15), and noting that $W = e^G$, $W(0) = 1.0$, gives

$$W(T) = \exp\left(\left[\frac{1}{N} \sum_k \mu_k - \frac{1}{2} \frac{1}{N^2} \sum_{k,m} \sigma_k \sigma_m \rho_{km} \right] T + \frac{1}{N} \sum_k \sigma_k (Z_k(T) - Z_k(0)) \right) \tag{2.18}$$

Let

$$\begin{aligned} \sigma_e^2 &= \frac{1}{N^2} \sum_{k,m} \sigma_k \sigma_m \rho_{km} \\ \mu_e &= \frac{1}{N} \sum_k \mu_k \ . \end{aligned} \tag{2.19}$$

We can then write equation (2.18) in the simpler form

$$W(T) = \exp\left((\mu_e - \frac{\sigma_e^2}{2})T + \frac{1}{N} \sum_k \sigma_k (Z_k(t) - Z_k(0)) \right) \ . \tag{2.20}$$

If we are only interested in the distribution of the rebalanced portfolio, and not pathwise comparison to the buy and hold portfolio (which is the market capitalization weighted index), then we can further simplify equation (2.20). Since the sum of normals is also normal, define a new Brownian process $\hat{Z}(t)$ with the property that

$$(\hat{Z}(t) - \hat{Z}(0)) \simeq \mathcal{N}(0, t) \ . \tag{2.21}$$

Now, we can rewrite equation 2.20

$$W(T) = \exp\left((\mu_e - \frac{\sigma_e^2}{2})T + \sigma_e(\hat{Z}(T) - \hat{Z}(0)) \right) \ . \tag{2.22}$$

The exact CDF for equation (2.22) is

$$\begin{aligned} \overbrace{CDF(W)}^{rebalance} &= \Phi\left(\frac{\log(W) - \mu'}{\sigma'} \right) \\ & \mu' = (\mu_e - \frac{\sigma_e^2}{2})T \\ & \sigma' = \sigma_e \sqrt{T} \\ & \Phi(\cdot) \ \text{ standard normal CDF } \ , \end{aligned} \tag{2.23}$$

with the properties

$$\begin{aligned} \overbrace{E[W(T)]}^{rebalance} &= \exp(\mu_e T) \\ \overbrace{Median[W_T]}^{rebalance} &= \exp\left((\mu_e - \frac{\sigma_e^2}{2})T \right) \\ \overbrace{Var[W_t]}^{rebalance} &= \exp(2\mu_e T)(e^{\sigma_e^2 T} - 1) \ . \end{aligned} \tag{2.24}$$

3 A special case: assets with identical properties

Now, let's look at what happens for the special case

$$\begin{aligned} \mu_i &= \mu \;\; ;\forall i \\ \sigma_i &= \sigma \;\; ;\forall i \\ \rho_{ij} &= \begin{cases} \rho \geq 0 & ; i \neq j \\ 1 & ; i = j \end{cases} . \end{aligned} \tag{3.1}$$

Note that even though all the assets have the same statistical parameters, this does not mean that all assets have the same value at $t = T$. These assets will follow different paths, since $Z_i \neq Z_j$.

3.1 Buy and hold

Assuming equation (3.1), then equations (2.6-2.7) and (2.10) become

$$\begin{aligned} \overbrace{W(T)}^{\textit{buy and hold}} &= \frac{1}{N} \sum_i \exp\Big((\mu - \sigma^2/2)T + \sigma(Z_i(T) - Z_i(0)) \Big) , \\ \overbrace{E[W_T]}^{\textit{buy and hold}} &= \exp(\mu T) . \end{aligned} \tag{3.2}$$

3.2 Rebalance to equal weight

From equations (2.19) and equation (3.1), we obtain

$$\sigma_e^2 = \sigma^2 \rho \Big(1 - \frac{1}{N}\Big) + \frac{\sigma^2}{N} \tag{3.3}$$

$$\mu_e = \mu , \tag{3.4}$$

which gives us

$$\overbrace{W(T)}^{\textit{rebalance}} = \exp\Big((\mu - \frac{\sigma_e^2}{2})T + \frac{1}{N} \sum_k \sigma(Z_k(T) - Z_k(0)) \Big) \tag{3.5}$$

$$\overbrace{E[W(T)]}^{\textit{rebalance}} = \exp(\mu T) \tag{3.6}$$

$$\overbrace{Median[W_T]}^{\textit{rebalance}} = \exp\Big((\mu - \frac{\sigma_e^2}{2})T \Big) \tag{3.7}$$

$$\overbrace{std[W_t]}^{\textit{rebalance}} = \exp(\mu_e T) \sqrt{(e^{\sigma_e^2 T} - 1)} . \tag{3.8}$$

Again, if we are not concerned with pathwise comparison with buy and hold, we can simplify the expression for W_T

$$\overbrace{W(T)}^{\textit{rebalance}} = \exp\Big((\mu - \frac{\sigma_e^2}{2})T + \sigma_e(\hat{Z}(T) - \hat{Z}(0)) \Big) . \tag{3.9}$$

3.3 Discussion: closed form results for model market

From Sections 3.1 and 3.2, we learn that

(i) The expected final wealth is the same for both buy and hold and rebalance to constant weight portfolios, i.e.

$$E[W_T] = e^{\mu T} \ .$$

(ii) For large N, then the effective volatility for the rebalance strategy becomes

$$\sigma_e \quad \simeq \quad \sigma\sqrt{\rho} \ \ ; \ \ N \to \infty \ ; \ \rho \geq 0 \ . \tag{3.10}$$

Compare equation (3.10) above to the properties of a single asset in our theoretical market (recall that all assets have the same drift μ and volatility σ values), in equation (2.9). If $\rho < 1$, then $\sigma_e < \sigma$.

This means that, compared to a single asset, the rebalanced portfolio has:

- the same expected final wealth;
- a larger median value for the final wealth;
- a smaller standard deviation.

Recall that the sum of log-normals is not, in general, log-normal, so we don't have any closed form results for the buy and hold strategy, except for the expected final wealth. However, since the buy-and-hold is simply a collection of single assets, which are never rebalanced, intuitively, we expect that we will not observe as large an improved median effect as we observe with the rebalancing strategy. We can say more about the buy and hold strategy after carrying out some simulations.

4 Numerical example

4.1 Data

Farago and Hjalmarsson (2023) use the CRSP dataset for 1973-2019 to determine the average stock return and volatility, and average pairwise correlation. The (rounded) values in Farago and Hjalmarsson (2023) are reported in Table 4.1. Other parameters for our theoretical market simulation also given in this table. The average stock volatility at 0.6235 is large compared to the historical average CRSP capitalization weighted index volatility of $\simeq .15 - .20$ (over 1926-2022).

The average pairwise correlation of 0.15 (see Table 4.1) is surprisingly (at least to me) low.

4.2 Single Stock Properties

Looking at the single stock assumptions from Table 4.1, and assuming each stock follows equation (2.1), we can determine some statistics for $(S_i(T)/S_i(0))$ (which are the same for all i). After $T = 30$ years, we have

$$\begin{aligned} \text{Mean}[S_i(T)/S_i(0)] &= 36.598 \\ \text{Median}[S_i(T)/S_i(0)] &= 0.107 \\ \text{Prob}[S_i(T)/S_i(0) < 1] &= 0.743 \end{aligned} \tag{4.1}$$

μ	.12
σ	.6235
ρ	.15
Initial wealth	1.0
Initial wealth in asset i	1/100
Number assets	100
Time T	30 years
Rebalance weight	1/100

TABLE 4.1: *Data for example. ρ is the average pairwise correlation, μ is the average arithmetic return, and σ is the average individual stock volatility, based on the Center for Research in Security Pricing (CRSP) stocks. Numbers are rounded, see Farago and Hjalmarsson (2023) for details. Time units are years.*

Note that the mean value of $(S_i(T)/S_i(0))$ after 30 years is 36.6, which is certainly impressive. However, the probability that $(S_i(T)/S_i(0)) < 1$ (i.e. the stock is a loser) is 74%. Yet the mean outcome is very large. This is an example of the *volatility drag* effect. Most of the outcomes are poor. The large mean value is due to some extreme, large, low-probability, high returns. These stocks certainly do not seem to be good individual investments.

Recall that the data in Table 4.1 are average values over all stocks in the CRSP database (Farago and Hjalmarsson, 2023).

4.3 Simulations

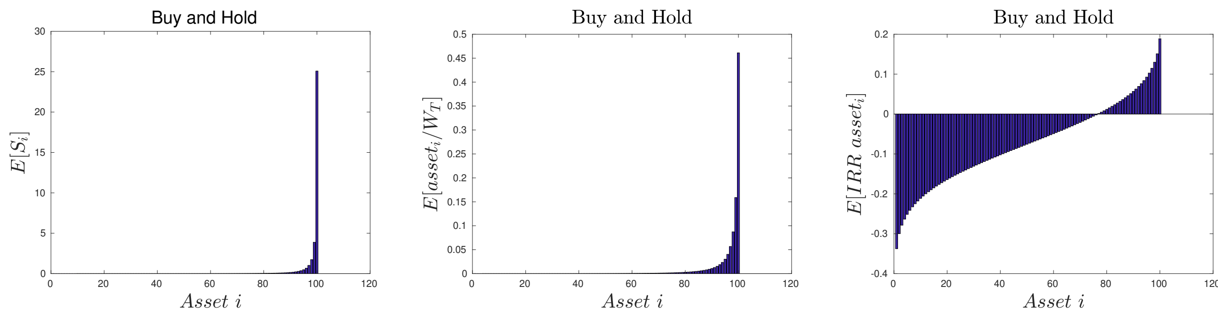
Since no closed form results are available for the buy and hold portfolio, or any pathwise comparison of buy and hold and rebalance, we use Monte Carlo methods to generate solutions to equation (2.1), for both strategies. Table 4.2 shows some summary statistics for the buy and hold, and the rebalanced portfolio. Note the rather large number of simulations, 80×10^6. This is based on examining the difference between the exact $E[W(T)]$ and the estimates from the simulations.

We can observe that both methods (to within MC error) have the same value for $E[W_T]$ as expected. However, the Median level of the final wealth for the rebalance strategy is more than twice the median of the buy and hold portfolio.

This is essentially due the fact the volatility drag is smaller for the rebalanced portfolio. An intuitive explanation is that rebalancing is a *"buy low, sell high"* contrarian strategy. This is also known more popularly as *volatility pumping.*

	$E[W_T]$	$std[W_T]$	$Median[W_T]$
	Buy and hold		
Simulation	36.405 (0.15)	677.45	6.649
Exact	36.598	N/A	N/A
	Rebalance		
Simulation	36.610 (0.02)	84.512	14.516
Exact	36.598	84.666	14.521

TABLE 4.2: *Summary statistics, 80×10^6 simulations. Data in Table 4.1 Standard MC errors, 95% confidence, shown in brackets. Initial investment $W(0) = 1.0$.*



(a) *Expected final amount in asset i.* $E[W(T)] = 36.6$.

(b) *Expected fraction of W_T in asset i along each path.*

(c) *Expected IRR of asset i.*

FIGURE 4.1: 80×10^6 *simulations, data in Table 4.1. Buy and hold case. Along each path, the assets are sorted in increasing order of final wealth. For example, asset* 100 *always refers to the asset which generated the most wealth (on any given path) followed by asset* 99*, and so on. From Figure 4.1(b) we can deduce that 95% of the final wealth is concentrated in just 16 (out of 100) assets. The best performing asset is expected to generate about 45% of the final wealth.*

More interesting results are shown in Figure 4.1. Along each stochastic path, the terminal wealth for each asset $i \in 1,...,100$ is obtained. Then these wealth values are sorted in increasing wealth order. Finally, all these sorted paths are averaged. In other words, asset $i = 100$ does not refer to the same asset, but, along each path, is the best performing asset. Asset $i = 99$ always refers to the second best performing asset, and so on. Figure 4.1(a) shows

$$E[S_i(T)] \ . \tag{4.2}$$

Recall that the mean value of the portfolio is about 36.6 and the mean value of the best performing asset along each path is about 25. This is quite impressive, since the initial allocation to each asset is 1/100.

Figure 4.1(b) shows the expected value of

$$E\Big[S_i(T)/W(T)\Big] \ , \tag{4.3}$$

which is the fraction of the total wealth in the i'th best performing asset, along each path. Figure 4.1(b) shows that the expected fraction of the total wealth which is held in the best performing asset, is about 45%. 95% of the final wealth is concentrated (on average) in just 16 (out of the original 100) assets.

Similarly, Figure 4.1(c) shows the internal rate of return for each asset (following the same sorting procedure as used previously). More precisely

$$E[IRR_i] \quad = \quad E\left[\frac{\log\big(S_i(T)/S_i(0)\big)}{T}\right] \ . \tag{4.4}$$

We can see that, along any path, we can expect 80 (out of 100) assets to have negative IRRs. The best performing asset has an expected IRR of 20% per year.

Figure 4.2 shows the pathwise CDF for the ratio

$$R = \frac{W_{rebal}(T)}{W_{buy+hold}(T)} \tag{4.5}$$

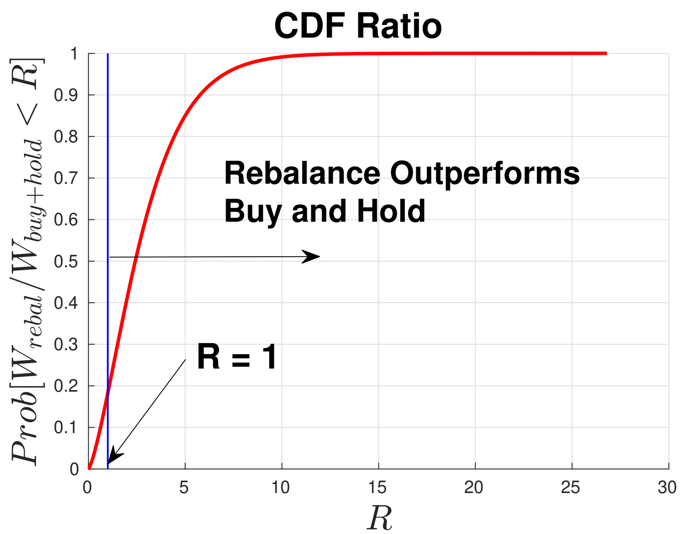


FIGURE 4.2: *CDF of* $R = W_{rebal}/W_{buy+hold}$. 80×10^6 *simulations, data in Table 4.1. Probability that rebalancing to equal weight outperforms buy and hold is* 0.82.

where $R > 1$ indicates that the rebalanced portfolio outperformed the buy and hold portfolio, which occurs with 82% probability.

Figure 4.3(a) appears to show that the rebalanced portfolio stochastically dominates the buy and hold portfolio. However, this is not rigorously true, since the curves cross at $(W_T, Prob) =$ (216,0.98), hence we have only partial stochastic dominance (van Staden et al., 2021). However, for practical purposes, we can say that the rebalanced portfolio is to preferred to the buy and hold portfolio, except possibly for the extreme right tail.

The density of the buy and hold strategy (see Figure 4.3(b)) is not precisely log normal, but we can fit the distribution to log-normal to get an intuitive feel for the distribution.

Let W_T be the terminal wealth for the buy and hold strategy. Then, we can estimate the arithmetic mean return from

$$\hat{\mu} \;=\; \log(E[W_T/W_0])/T; \tag{4.6}$$

Of course, in the limit as the number of simulations becomes large, $\hat{\mu} \rightarrow \mu$.

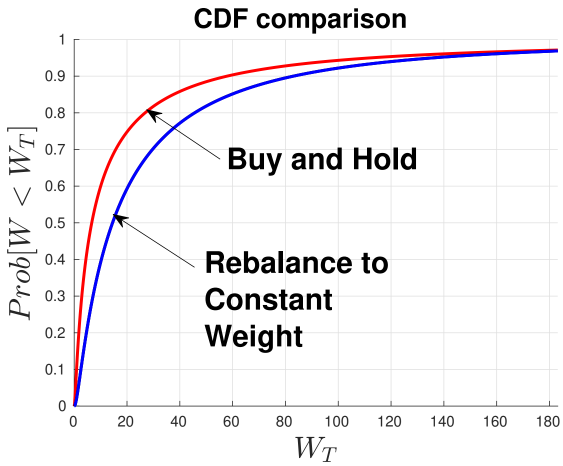
We can fit the effective volatility $\hat{\sigma}$ in two ways

$$\hat{\sigma} \;=\; std(\log(W_T/W_0))/T \tag{4.7}$$

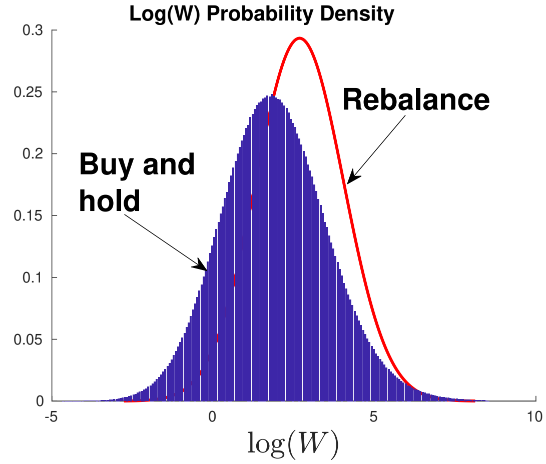
$$\hat{\sigma} \;=\; \sqrt{2\Big(\hat{\mu}T - Median[\log(W_T/W_0)]\Big)/T}\ . \tag{4.8}$$

Of course, these two estimates should give the same result if the distribution was log-normal, but this will not be true in our case, since the distribution of W_T (buy and hold) is not exactly log-normal. Table 4.3 shows the two estimates for $\hat{\sigma}$. The buy and hold $\hat{\sigma}$ is larger than the rebalance effective volatility σ_e but much smaller than the single stock volatility σ.

Table 4.4 shows the fitted volatility for the buy and hold portfolio as a function of investment horizon T. At $T = 10$ years, the buy and hold fitted volatility is only slightly larger than the rebalance σ_e. This is reflected in the $Median[W_T]$ for both strategies. For larger T, $\hat{\sigma}$ increases (while σ_e remains constant). This shows up as a larger difference in $Median[W_T]$ for the two



(a) *Comparison of CDFs, rebalance vs. buy and hold.*



(b) *Density functions of* $\log(W_T)$*, comparison of buy and hold and rebalance.*

FIGURE 4.3: *CDF of* $R = W_{rebal}/W_{buy+hold}$. 80×10^6 *simulations, data in Table 4.1. Comparison of CDFs, rebalance to constant weight and buy and hold. Rebalancing partially stochastically dominates buy and hold, with the curves crossing at* $W_T = 216$*. Rebalancing dominates buy and hold with for probabilities* < 0.98.

	Effective Volatility
Buy and hold	
$\hat{\sigma}$ std $[\log(W_T)]$ (4.7)	.3001
$\hat{\sigma}$ median $[\log(W_T)]$ (4.8)	.3367
Rebalance	
σ_e (2.19)	.24824
Single Stock	
σ	.6235

TABLE 4.3: *Exact effective volatility, rebalance. Approximate fitted volatilities, buy and hold. Single stock volatility also shown. Data in Table 4.1*

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strategies. The effect of different time horizons is also shown in Figure 4.4. For short time horizons (e.g. $T = 10$ years), the probability of pathwise outperformance for rebalance versus buy and hold is only 0.64, while for $T = 60$ years, the rebalancing strategy outperforms buy and hold with 94% probability.

T	Buy and hold $\hat{\sigma}$	Buy and Hold: Median$[W_T]$	Rebalance Median$[W_T]$
10	.274	2.28	2.44
30	.337	6.65	14.52
60	.392	13.3	210.9

TABLE 4.4: *Fitted effective volatilities* $\hat{\sigma}$*, buy and hold, using equation (4.8).* $\sigma_e = .248$ *rebalance. Data in Table 4.1*

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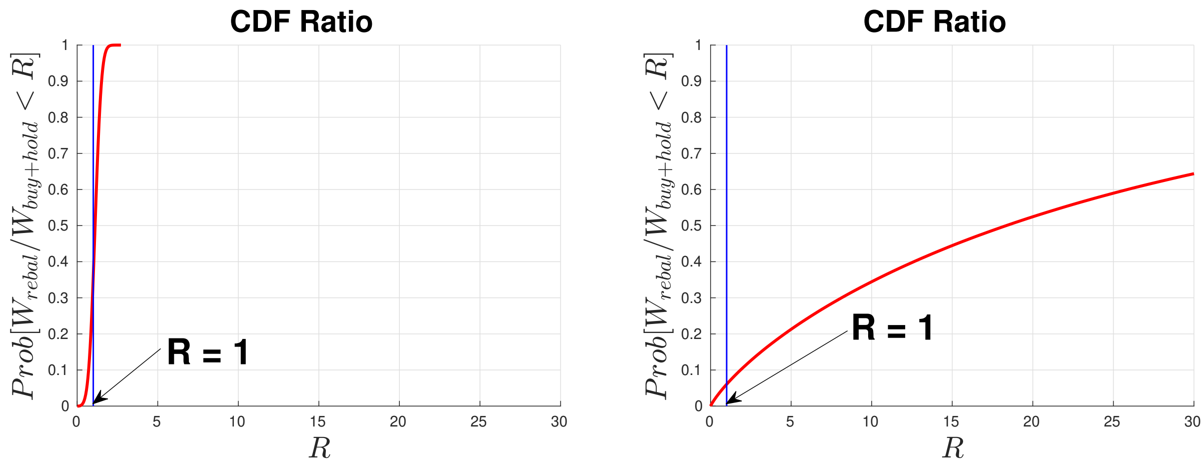

(a) $T = 10$ *years.* $Prob[W_{rebal} > W_{buy+hold}] = 0.64$.

(b) $T = 60$ *years.* $Prob[W_{rebal} > W_{buy+hold}] = 0.94$.

FIGURE 4.4: *CDF of* $R = W_{rebal}/W_{buy+hold}$. 80×10^6 *simulations, data in Table 4.1. Effect of changing the base case investment horizon* $T = 30$. *Compare with Figure 4.2.*

Finally, Table 4.5 shows the effect of varying ρ and σ. Figure 4.5(b) shows the result obtained by increasing ρ to $\rho = 0.5$ and decreasing the single stock volatility to $\sigma = 0.40$. In this case, the CDF curves for the two strategies essentially overlap, indicating that there is no particular advantage to rebalancing.

	σ	ρ	$Prob[W_{rebalance} > W_{buy+hold}]$
Base Case	.6235	.15	0.82
$\sigma \downarrow$	.40	.15	0.67
$\rho \uparrow$	.6235	.50	0.72
$\sigma \downarrow, \rho \uparrow$	.40	.50	0.61

TABLE 4.5: 80×10^6 *simulations. Data in Table 4.1. Effect of varying* σ, ρ.

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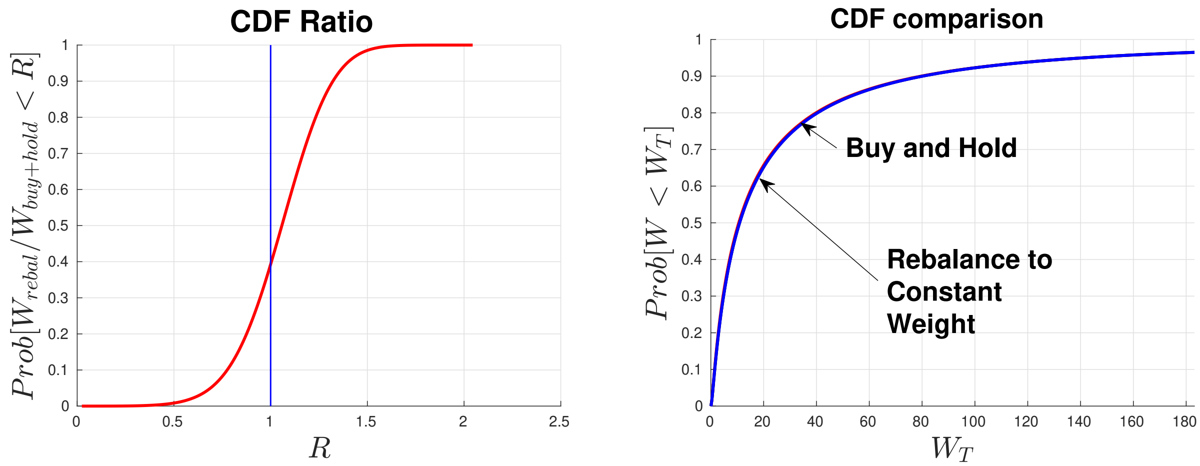
5 Summary

Using the single stock properties in Table 4.1 we observe that, for $T = 30$ years

- there is a significant concentration effect for the buy and hold portfolio (which is the capitalization weighted index in this market). Only a few stocks which have large gains account for most of the expected terminal wealth;
- for the base case, the rebalancing strategy partially stochastically dominates the buy and hold strategy, hence would be preferred by most investors;
- these results are very sensitive to the parameters in our tests. The superiority of the rebalance policy essentially disappears for (i) short time horizons (ten years) (ii) larger pairwise stock correlation and smaller single stock volatilities.



(a) $T = 30$ *years.* $Prob[W_{rebal} > W_{buy+hold}] = 0.61$.

(b) $T = 30$ *years. Comparison of CDFs: buy and hold, rebalance visually overlap.*

FIGURE 4.5: $T = 30$ *years.* 80×10^6 *simulations, data in Table 4.1, with the exception that* $\rho = 0.5$, $\sigma = 0.40$.

6 Conclusion

The main empirical results in Bessembinder (2018) and Farago and Hjalmarsson (2023) are that

- most stocks are not competitive with T-bills, over their lifetime;
- a small, equal-weighted portfolio has a high probability of outperforming a broad cap-weighted index.

Farago and Hjalmarsson (2023) put forward the properties of a stylized portfolio of stocks with identical properties, to provide an explanation for these empirical results.

It is indeed intriguing to see that a simple model, which assumes that stocks follow geometric Brownian motion (GBM), results in a very concentrated market over time. This means that the rebalancing to equal weight strategy partially stochastically dominates the buy and hold portfolio. However, if we change the parameters in this stylized market, this partial dominance can become insignificant.

What do these results mean for an investor? The empirical fact that equal-weighted portfolios tend to dominate capitalization weighted indexes seems to be a robust empirical fact. However, this is only true over the long term. This is especially evident in the recent performance of the $S\&P$ 500, which has been dominated by the performance of the *magnificent seven* stocks. The fact that the dominance of the rebalanced portfolio appears only in the long term, is consistent with our stylized model.

It is also clear that the long term volatility of the $S\&P$ 500 index is in the range of $.15 - .20$, which is much smaller than the fitted estimates from our model market. This is probably due to a number of effects. As stocks age, with larger capitalization weight in the index, volatilities seem to decrease (Farago and Hjalmarsson, 2023). We can also imagine that the pairwise correlations between these high performing, large capitalization stocks also tends to increase.

There is also an effective rebalancing which occurs in an index. This is due to poorly performing stocks being dropped from the index, and replaced with new stocks.[6] In addition, all stocks do

[6]For example, over the period 1970-2021, there were 1194 stock deletions in the S&P 500 index (Arnott and Brightman, 2023).

not pay the same proportional dividends. Investors may choose to invest/spend the dividends, or reinvest the dividends in the total index, which will cause departures from pure buy and hold.

Note as well that, in the event that our basket of stocks has different drift rates (arithmetic expected return), the buy and hold strategy will eventually consist primarily of the stocks which have the largest expected returns. Of course, this will eventually dominate a rebalanced portfolio. However, this may take a very long time. This is perhaps not so relevant in practice, since a given stock will almost certainly not consistently outperform all the other stocks for long time periods.

It is simplistic to dismiss the performance of equal-weight indexes as simply due to the small capitalization effect. Rebalancing seems to be a significant factor in the observed performance. However, the rebalancing effect does not fully explain what is going on here. The rebalancing effect produced in the model market is too large to explain the observed capitalization weighted index behaviour.

The bottom line

- investors should probably have some allocation to equal weight indexes. It is, of course, important to minimize the tax consequences of rebalancing. US ETFs can often avoid taxes on rebalancing[7]. Other countries permit holding ETFs in non-taxable accounts;

- the model market, which magnifies the rebalancing effect, shows that the equal weight outperformance is only significant for long term investors (i.e. > 10 years), and if the constituent stocks have low pairwise correlations and high volatilities.

Recall that at one point Nokia comprised 70% of the Finnish stock market, and Nortel was 35% of the TSE 300 composite index. The model market suggests that these sorts of extreme concentrations are not unlikely. However, this does not end well.

[7] `https://www.bloomberg.com/graphics/2019-etf-tax-dodge-lets-investors-save-big/`

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