Optimal Asset Allocation for Retirement Saving: Deterministic vs. Time Consistent Adaptive Strategies

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Abstract

We consider optimal asset allocation for an investor saving for retirement. The portfolio contains a bond index and a stock index. We use multi-period criteria and explore two types of strategies: *deterministic* strategies are based only on the time remaining until the anticipated retirement date, while *adaptive* strategies also consider the investor's accumulated wealth. The vast majority of financial products designed for retirement saving use deterministic strategies (e.g. target date funds). In the deterministic case, we determine an optimal open loop control using mean-variance criteria. In the adaptive case, we use time consistent mean-variance and quadratic shortfall objectives. Tests based on both a synthetic market where the stock index is modeled by a jump diffusion process and also on bootstrap resampling of long-term historical data show that the optimal adaptive strategies significantly outperform the optimal deterministic strategy. This suggests that investors are not being well-served by the strategies currently dominating the marketplace.

Keywords: finance, time consistent mean-variance, quadratic shortfall, dynamic asset allocation, jump diffusion, resampled backtests, deterministic strategy
JEL codes: G11, G22

1 Introduction

Saving for retirement is one of the most important financial tasks faced by individuals. The total value of retirement assets in the U.S. at the end of 2016 was about $25 trillion (ICI, 2017), exceeding U.S. GDP for that year by around 35%. More than 60% of these assets were held in individual retirement accounts and defined contribution (DC) pension plans, reflecting the long-term decline in traditional defined benefit (DB) plans. The fundamental reason underlying this trend is that DB plans are seen as a high risk liability for many organizations, and the risk is being transferred to employees through vehicles such as DC plans.

In a DC plan, the employee contributes a fraction of her salary to a tax-advantaged account. This amount is often matched by the employer. The employee is responsible for managing the investments in the account. An accumulation period lasting 30 years would not be unusual, followed by a de-accumulation (retirement) phase of another 20 years, so the employee could end up managing a significant portfolio for 50-60 years. This makes participants in DC plans truly long-term investors.

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This study deals with the accumulation phase. Several observers have expressed doubts about the ability of individuals to adequately save for retirement (e.g. Benartzi and Thaler, 2001; 2007; Choi et al., 2004). Three general concerns are (i) whether individuals enrol in savings plans, (ii) if they contribute enough, and (iii) whether they choose appropriate investments. With respect to the first two concerns, significant progress has been achieved through *automatic enrolment* and *automatic escalation*. Automatic enrolment exploits the tendency for individuals to stick with the status quo. Employees are put into a plan by default, while having the choice to easily opt out, instead of having to actively choose to participate. Firms adopting automatic enrolment have seen very strong increases in plan participation rates (Madrian and Shea, 2001; Benartzi and Thaler, 2007). Automatic escalation involves increasing contribution rates over time, as the employee's salary goes up. The Pension Protection Act of 2006 encouraged firms to adopt both automatic enrolment and automatic escalation, and by 2011 over half of firms offering 401(k) plans were doing so (Benartzi and Thaler, 2013).

Offering automatic enrollment entails specifying a default investment option, the third concern noted above. This asset allocation issue is the focus of this study. More than a decade ago, it was common to offer a low-risk default choice such as a money market savings account (Choi et al., 2004). The obvious concern this raised was whether investors could realize high enough returns to accumulate sufficient retirement funds, without taking on more risk. Target date funds (TDFs, also known as lifecycle funds) have become a significant component of the industry's response to doubts that individual investors would be capable of appropriately managing the risk of their retirement savings portfolios. The buyer of a TDF specifies a target date, normally the anticipated retirement date. The most basic TDF consists of a bond index and an equity index. A typical TDF specifies a *glide path*, which determines the fraction of the total portfolio that is invested in the equity index (with the remainder in the bond index) as a function of time. The Pension Protection Act of 2006 permitted TDFs to be used as default investment options in DC plans. Total assets invested in U.S. TDFs have increased dramatically over the past decade, reaching $887 billion at the end of 2016, up from $70 billion in 2005 (ICI, 2017, Figure 7.25). The single largest provider of TDFs is Vanguard, with total net assets of about $280 billion at the end of 2016.[1] Vanguard reports that:

> Nine in 10 plan sponsors offered target-date funds at year-end 2016, up over 50% compared with year-end 2007. Nearly all Vanguard participants (97%) are in plans offering target-date funds. 72% of all participants use target-date funds. Two-thirds of participants owning target-date funds have their entire account invested in a single target-date fund. 46% of all Vanguard participants are wholly invested in a single target-date fund, either by voluntary choice or by default (Vanguard, 2017, p. 3).

Moreover, at the end of 2016 83% of Vanguard DC plans specified TDFs as the default investment choice (Vanguard, 2017, Figure 62). Given the propensity of participants to stay with default options, continued strong growth of TDFs appears very likely over the next few years.

The prototypical TDF glide path has a high allocation to stocks during the early years of the accumulation phase. The equity allocation is decreased (and the bond allocation increased) as the time remaining to the target date declines. The underlying rationale is that with many years to retirement, the investor can take on more risk since there is time to recover from adverse market returns. However, as the target date nears, the portfolio is weighted more to bonds as protection against market downturns. This seems to be an intuitively appealing strategy.

The vast majority of TDFs use a *deterministic* glide path. In other words, the bond-stock split is only a function of the time remaining until the target date. This contrasts with an *adaptive* strategy,

[1] See `http://news.morningstar.com/cover/videocenter.aspx?id=788811`.

where the asset allocation can be a function of the time remaining and the accumulated wealth so far. In control terminology, a deterministic strategy can be interpreted as open loop control, while an adaptive strategy is a closed loop control. Adaptive strategies have not received much attention to date. One exception is Basu et al. (2011), who consider a type of adaptive strategy using heuristic adjustments based on cumulative investment performance. In particular, they propose strategies that are 100% allocated to equities for a lengthy period, e.g. 20 years. Subsequently, the asset allocation can be switched to 80% equity and 20% in fixed income if overall performance has been satisfactory relative to a specified target; otherwise the portfolio remains completely invested in equities. Portfolio performance is then re-evaluated each year, with similar adjustments based on cumulative performance relative to target. While the adaptive strategies we consider here are similar in spirit, they are based on more robust methods of stochastic optimal control, in contrast to the *ad hoc* adjustments proposed by Basu et al. (2011).

We restrict attention here to an investment portfolio containing a stock and bond index. We model the real (inflation-adjusted) stock index as following a jump diffusion, with the jumps having a double exponential distribution (Kou, 2002; Kou and Wang, 2004). The jump component allows for skewed and leptokurtic returns, and the double exponential distribution fits equity index returns better than a model with lognormally distributed jumps (Ramezani and Zeng, 2007). The diffusion component is simply geometric Brownian motion with constant volatility. An obvious extension would be to allow for random changes in volatility over time, but previous work has shown that mean-reverting stochastic volatility effects are negligible for long-term investors (Ma and Forsyth, 2016), so we use the simpler formulation here. We fit the parameters of the jump diffusion model to 90 years of market data.

We develop adaptive strategies based on two objective functions: time consistent dynamic (multi-period) mean-variance (MV) and expected quadratic shortfall. In the MV case, we consider strategies which minimize the variance of real terminal wealth for a given specified expected value of real terminal wealth, with the addition of a time consistent constraint. In the expected quadratic shortfall case, we base our strategy on minimizing the expected quadratic loss of the terminal wealth with respect to a fixed real target final wealth.

This means we are concentrating on the risk of the *outcome*, rather than the risk of the *process* along the way. As an example of process risk, some would argue that we should be concerned with the volatility of the investment portfolio throughout the entire investment period. However, adding constraints on the local volatility will lead to sub-optimal results compared with fixing attention on the terminal wealth distribution. We contend that focusing on the long-term investment goal is appropriate for retirement savings. However, while we focus on outcome risk, we implicitly take process risk into account to some extent through constraints such as not allowing any use of leverage.

Investors saving for retirement are primarily interested in accumulating assets in order to fund a reasonable standard of living post-retirement, at minimal risk. Hence a (real) target-based final wealth strategy seems appropriate in this context (Vigna, 2014; Menoncin and Vigna, 2017). Note that target final wealth based objective functions are entirely different from equilibrium-based, instantaneous return constrained objective functions such as those considered by He and Jiang (2017).

We develop MV optimal deterministic strategies, as well as our two time consistent adaptive strategies. We provide two types of extensive comparisons between them. First, we use a *synthetic market* that relies on Monte Carlo simulations which assume that the stock and bond indexes follow the models with constant parameters fit from the entire historical time series. Second, we compare the strategies using bootstrap resampling of the actual historical data (Politis and Romano, 1994; Cogneau and Zakalmouline, 2013; Dichtl et al., 2016). We emphasize that all strategies enforce realistic constraints, e.g. no short sales or leverage, no trading if insolvent, discrete rebalancing, etc. This is important because unconstrained dynamic MV strategies may involve the use of highly levered portfolios (Lioui and Poncet, 2016).

Our main results are as follows:

- For a lump sum investment in the synthetic market with continuous rebalancing, a constant proportion strategy is superior in the MV sense to any deterministic glide path.
- For a discretely rebalanced long-term portfolio with regular periodic contributions, the optimal deterministic strategy gives only a very slight improvement (under MV criteria) over a constant proportion strategy.
- The risk-reward tradeoff given by the optimal deterministic strategy for a portfolio with regular contributions does not improve much if the portfolio is rebalanced more often than annually. This implies that infrequent rebalancing is not costly in terms of MV criteria, while offering the benefits of lower trading costs.
- The optimal adaptive strategies significantly outperform the deterministic strategies in terms of the median and standard deviation of final wealth compared to the optimal deterministic strategy having the same expected final wealth. The probabilities of shortfall for a wide range of terminal wealth values are also substantially reduced for the adaptive strategies compared to the deterministic strategies.
- Our strategies are based on very parsimonious models for real (i.e. inflation-adjusted) stock and bond indexes. We test the strategy on bootstrapped resamples of the historical market returns, and we find that our adaptive strategies are robust in the real historical market. This is a rather satisfying result: for long-term investors, an adaptive strategy based on a parsimonious model of real stock and bond returns is superior to deterministic glide path strategies.

Our overall conclusion is that the current deterministic strategies used in most TDFs are sub-optimal relative to adaptive strategies. While it is unrealistic to assume that individual investors could determine optimal adaptive strategies themselves, it certainly is possible for sophisticated financial intermediaries to provide them to their clients.

2 Formulation

For simplicity we assume that there are only two assets available in the financial market, namely a risky asset and a risk-free asset. In practice, the risky asset would be a broad market index fund. We believe that for long-term investors, the major asset allocation decision is the stock-bond split. We will use very parsimonious stochastic models, with a small number of parameters which are calibrated to long run historical data. This approach minimizes the problem of strongly varying asset allocations over time, due to calibration instability. We will verify that the strategies are robust by carrying out bootstrap resampling tests using the historical data.

The investment horizon is T. S_t and B_t respectively denote the *amounts* invested in the risky and risk-free assets at time t, $t \in [0, T]$. In general, these amounts will depend on the investor's strategy over time, including contributions, withdrawals, and portfolio rebalances, as well as changes in the unit prices of the assets. The investor can control all of these factors except for the unit prices. To clarify our assumptions regarding asset price dynamics, suppose for the moment that the investor does not take any action with respect to the controllable factors. We refer to this as the absence of control. It implies that all changes in S_t and B_t result from changes in asset prices. In this case, we assume that S_t follows a jump diffusion process. Let $t^- = t - \epsilon, \epsilon \to 0^+$, i.e. t^- is the instant of time before t, and let ξ be a random number representing a jump multiplier. When a jump occurs, $S_t = \xi S_{t^-}$. Allowing discontinuous jumps allows us to explore the effects of severe market crashes on the risky asset holding. We assume that ξ follows a double exponential distribution (Kou, 2002; Kou and Wang, 2004). If a jump occurs, p_{up} is the probability of an upward jump, while $1 - p_{up}$ is the chance of a downward jump. The density function for $y = \log(\xi)$ is

$$f(y) = p_{up}\eta_1 e^{-\eta_1 y}\mathbf{1}_{y\geq 0} + (1 - p_{up})\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}. \tag{2.1}$$

For future reference, note that

$$\begin{aligned} E[y = \log \xi] &= \frac{p_{up}}{\eta_1} - \frac{(1 - p_{up})}{\eta_2}, \quad E[(\log \xi)^2] = \frac{2p_{up}}{\eta_1^2} + \frac{2(1 - p_{up})}{\eta_2^2} \\ E[\xi] &= \frac{p_{up}\eta_1}{\eta_1 - 1} + \frac{(1 - p_{up})\eta_2}{\eta_2 + 1}, \\ E[(\xi - 1)^2] &= \frac{p_{up}\eta_1}{\eta_1 - 2} + \frac{(1 - p_{up})\eta_2}{\eta_2 + 2} - 2\left(\frac{p_{up}\eta_1}{\eta_1 - 1} + \frac{(1 - p_{up})\eta_2}{\eta_2 + 1}\right) + 1. \end{aligned} \tag{2.2}$$

In the absence of control, S_t evolves according to

$$\frac{dS_t}{S_{t^-}} = (\mu - \lambda E[\xi - 1])\, dt + \sigma\, d\mathcal{Z} + d\left(\sum_{i=1}^{\pi_t}(\xi_i - 1)\right), \tag{2.3}$$

where μ is the (uncompensated) drift rate, σ is the volatility, $d\mathcal{Z}$ is the increment of a Wiener process, π_t is a Poisson process with positive intensity parameter λ, and ξ_i are i.i.d. positive random variables having distribution (2.1). Moreover, ξ_i, π_t, and Z are assumed to all be mutually independent.

As an aid to carrying out algebraic manipulations, we can write (2.3) more informally as

$$\frac{dS_t}{S_{t^-}} = (\mu - \lambda E[\xi - 1])\, dt + \sigma\, d\mathcal{Z} + (\xi - 1)\, d\mathcal{Q}, \tag{2.4}$$

where $d\mathcal{Q} = 1$ with probability $\lambda\, dt$ and $d\mathcal{Q} = 0$ with probability $1 - \lambda\, dt$.

In the absence of control, we assume that the dynamics of the amount B_t invested in the risk-free asset are

$$dB_t = rB_t\, dt, \tag{2.5}$$

where r is the (constant) risk-free rate.

Remark 2.1 (Parsimonious Model). *Equations* (2.4)-(2.5) *are very simple specifications that assume both constant equity market volatility and constant real interest rate. In other contexts, these specifications would be overly simplistic. For example, if we were concerned with valuation or hedging of contracts with embedded optionality, it would be important to incorporate stochastic volatility effects. However, our setting involves long-term asset allocation, with infrequent rebalancing. A typical mean-reverting stochastic volatility specification has little impact in this context, since the duration of volatility shocks is typically shorter than the rebalancing period (Ma and Forsyth, 2016). As for the constant interest rate assumption, recall that we are concerned with real bond indexes. Such indexes have quite low volatility, particularly if the underlying instrument is short-term in nature. We utilize equations* (2.4)-(2.5) *to determine the optimal strategy in the synthetic market. We apply this strategy to both the synthetic market and also to real bootstrapped data, with similar statistical results. In essence then, equations* (2.4)-(2.5) *seem sufficient for generating an adaptive strategy which is superior to a deterministic strategy.*

We define the investor's total wealth at time t as

$$\text{Total wealth } \equiv W_t = S_t + B_t. \tag{2.6}$$

Given a specified expected value of terminal wealth $E[W_T]$, the investor wants to minimize the risk of achieving this expected terminal wealth. We impose the constraints that shorting stock and using leverage (i.e. borrowing) are not permitted, which would be typical of a retirement savings account.

3 Deterministic Glide Paths

Let p denote the fraction of total wealth that is invested in the risky asset, i.e.

$$p = \frac{S_t}{S_t + B_t}. \tag{3.1}$$

A *deterministic* glide path restricts the admissible strategies to those with $p = p(t)$, i.e. the optimal strategy cannot take into account the actual value of W_t at any time. Clearly this is a very restrictive assumption, but it is commonly used in TDFs. Although a constant proportion strategy can be seen as a special case of a deterministic glide path where $p(t) = \mathit{const.}$, it is simpler here for expository reasons to reserve the label "deterministic glide path" for cases where $p(t)$ is time-varying.

3.1 Lump Sum Investment with Continuous Rebalancing

To gain some intuition about deterministic strategies, we consider first a simple case with a lump sum initial investment and no further cash injections or withdrawals. We also assume here that the portfolio is continuously rebalanced. Under these conditions, we can derive:

Proposition 3.1 (Inefficiency of glide path strategies for lump sum investments)**.** *Consider a market with two assets following the processes* (2.4) *and* (2.5)*. Suppose we invest a lump sum* W_0 *at* $t = 0$ *in a continuously rebalanced portfolio using a deterministic glide path strategy* $p = p(t)$*, where* p *is the fraction of total wealth invested in the risky asset. Also consider a strategy with a constant proportion* p^* *invested in the risky asset, where*

$$p^* = \frac{1}{T}\int_0^T p(s)\, ds. \tag{3.2}$$

Then:

(i) the expected value of the terminal wealth is the same for both strategies; and

(ii) the standard deviation of terminal wealth for the glide path strategy cannot be less than that of the constant proportion strategy.

Proof. Equations (2.4) and (2.5) imply

$$\begin{aligned} \frac{dW_t}{W_{t^-}} &= p(t)\left(\frac{dS_t}{S_{t^-}}\right) + (1 - p(t)\ )\left(\frac{dB_t}{B_t}\right) \\ &= [p(t)(\mu - r) + r]\ dt - \lambda p(t) E[\xi - 1]\, dt + p(t)\sigma\, d\mathcal{Z} + p(t)(\xi - 1)\, d\mathcal{Q}. \end{aligned} \tag{3.3}$$

Letting $\overline{W}_t = E[W_t]$ and noting that $p(t)$ is deterministic, we have

$$d\overline{W}_t = [p(t)(\mu - r) + r]\, \overline{W}_t \, dt \tag{3.4}$$

and

$$\overline{W}_T = E[W_T] = W_0 e^{[p^*(\mu - r) + r]T}, \tag{3.5}$$

where p^* is defined in equation (3.2). Write equation (3.3) as

$$\frac{dW_t}{W_{t^-}} = \hat{\mu}\, dt + p(t)\sigma \, d\mathcal{Z} + p(t)(\xi - 1)\, d\mathcal{Q}, \tag{3.6}$$

where $\hat{\mu} = [p(t)(\mu - r) + r] - \lambda p(t) E[\xi - 1]$. Let $G_t = W_t^2$. From equation (3.6) and Itô's Lemma for jump processes,

$$\frac{dG_t}{G_{t^-}} = \left[2\hat{\mu} + (p(t)\sigma)^2\right] \, dt + 2p(t)\sigma \, dZ + \left[p(t)^2(\xi - 1)^2 + 2p(t)(\xi - 1)\right] \, d\mathcal{Q}. \tag{3.7}$$

Let $\overline{G}_t = E[G_t] = E[W_t^2]$. Equation (3.7) and the fact that $p(t)$ is deterministic imply

$$\begin{aligned} \frac{d\overline{G}_t}{\overline{G}_t} &= \left[2\hat{\mu} + (p(t)\sigma)^2\right] \, dt + \left(\lambda p(t)^2 E\left[(\xi - 1)^2\right] + 2\lambda p(t) E\left[(\xi - 1)\right]\right) dt \\ &= \left(2\left[p(t)(\mu - r) + r\right] + p(t)^2 \sigma_e^2\right) \, dt \end{aligned} \tag{3.8}$$

where $\sigma_e^2 = \sigma^2 + \lambda E\left[(\xi - 1)^2\right]$. This in turn gives

$$\overline{G}_T = G_0 \exp\left(2\left[p^*(\mu - r) + r\right] T + \sigma_e^2 \int_0^T p(s)^2 \, ds\right), \tag{3.9}$$

or

$$E[W_T^2] = (E[W_T])^2 \exp\left[\sigma_e^2 \int_0^T p(s)^2 \, ds\right]. \tag{3.10}$$

From $Var[W_T] = E[W_T^2] - (E[W_T])^2$, we obtain

$$std[W_T] = E[W_T]\left(\exp\left[\sigma_e^2 \int_0^T p(s)^2 \, ds\right] - 1\right)^{1/2} \tag{3.11}$$

where $std[\,\cdot\,]$ denotes standard deviation. By the Cauchy-Schwartz inequality

$$(p^*)^2\, T \leq \int_0^T p(s)^2 \, ds, \tag{3.12}$$

and Proposition 3.1 follows immediately. □

This proposition suggests that deterministic glide path strategies may have been oversold. A similar result for the geometric Brownian motion case (i.e. no jumps) was noted by Graf (2017). Furthermore, several authors have suggested that deterministic glide path strategies do not appear to offer many advantages based on Monte Carlo and historical simulations. For example, Poterba et al. (2009) simulate scenarios involving periodic contributions based on a sample of household earnings trajectories and investment returns based on resampled annual returns. They find that allocating wealth to assets based on age does not outperform a simple constant proportion strategy, noting that

The similarity of the retirement wealth distributions from the life-cycle portfolios, and from strategies that allocate a constant portfolio share to equities, is one of the central findings of our analysis. This result calls for further work to evaluate the extent to which life-cycle strategies offer unique opportunities for risk reduction relative to simpler strategies that allocate a constant fraction of portfolio assets to equities at all ages. (Poterba et al., 2009, p. 38)

Basu et al. (2011) and Esch and Michaud (2014) also find that glide paths do not seem to provide significant benefits in comparison to simpler fixed proportion strategies. Under some simplified assumptions, Proposition 3.1 shows that this result must hold: for any glide path, there is an equivalent constant weight strategy that offers the same expected final wealth at equal or lower risk. It is not surprising, then, to find that this is approximately correct in more complex and realistic simulations.

Along somewhat different lines, Arnott et al. (2013) simulate an inverse glide path which starts out with a low equity allocation that is increased over time. Their simulations show that this results in, if anything, better performance than the standard glide path which reduces equity exposure over time. Arnott et al. attribute this counterintuitive result to the effect of contributions on portfolio size over time. The standard glide path is most heavily invested in equities early on when the portfolio is fairly small. It does not benefit as much in monetary terms from high equity returns as the inverse glide path strategy, which has higher wealth when it is most exposed to equities. Basu et al. (2011) make a similar point, noting that the standard glide path approach can perform poorly because switching out of equities into bonds at a time when accumulated wealth (and possibly also contributions, if these are a fixed percentage of salary which has increased over time) is relatively large, "the investor may be foregoing the opportunity to earn higher returns on a larger sum of money invested" (Basu et al., 2011, p. 84). However, we point out that even in the case of a single lump sum contribution, the standard glide path intuition fails. Note that $\int_0^T p(s)\,ds = \int_0^T p(T-s)\,ds$ and $\int_0^T [p(s)]^2\,ds = \int_0^T [p(T-s)]^2\,ds$, so by equations (3.5) and (3.11) the glide path results are the same in this case if we reverse the strategy. In other words, if our glide path starts with a high allocation to stocks and finishes with a low allocation to stocks, we can achieve exactly the same MV result in terms of final wealth by beginning with a low equity allocation and ending with a high equity allocation.

3.2 Discrete Rebalancing and Periodic Contributions

The results in Section 3.1 are useful for gaining some intuition about the performance of glide path strategies, but the assumptions of no cash injections and continuous rebalancing are unrealistic. We now consider the implications of periodic cash injections and discrete portfolio rebalancing.

Let the inception time of the investment be $t_0 = 0$. We consider a set $\mathcal{T}$ of pre-determined *rebalancing times*,

$$\mathcal{T} \equiv \{t_0 = 0 < t_1 < \cdots < t_M = T\}. \tag{3.13}$$

For simplicity, we specify $\mathcal{T}$ to be equidistant with $t_i - t_{i-1} = \Delta t = T/M$, $i = 1, \ldots, M$. At each rebalancing time t_i, $i = 0, 1, \ldots, M-1$, the investor injects an amount of cash q_i into the portfolio and then rebalances the portfolio. At $t_M = T$, the portfolio is liquidated. Let $t_i^- = t_i - \epsilon$ ($\epsilon \to 0^+$) be the instant before rebalancing time t_i, and $t_i^+ = t_i + \epsilon$ be the instant after t_i. Let $p(t_i^+) = p_i$ be the fraction in the risky asset at t_i^+. This fraction is deterministic, so we can find some simple recursive expressions for the mean and variance of terminal wealth at $t = t_M$.

Similarly, let $S_i^+ = S_{t_i^+}$, $S_i^- = S_{t_i^-}$, $B_i^+ = B_{t_i^+}$, and $B_i^- = B_{t_i^-}$. From equations (2.4) and (2.5) we obtain

$$\begin{aligned} E\left[S_{i+1}^-\right] &= E\left[S_i^+\right]\exp[\mu\Delta t] \\ E\left[B_{i+1}^-\right] &= E\left[B_i^+\right]\exp[r\Delta t]. \end{aligned} \tag{3.14}$$

Since $W_i^- = S_i^- + B_i^-$,

$$\begin{aligned} W_i^+ &= W_i^- + q_i = S_i^- + B_i^- + q_i \\ E\left[W_i^+\right] &= E\left[S_i^-\right] + E\left[B_i^-\right] + q_i. \end{aligned} \tag{3.15}$$

Then

$$\begin{aligned} S_i^+ &= p_i W_i^+ \\ B_i^+ &= (1-p_i) W_i^+ \\ E\left[S_i^+\right] &= p_i E\left[W_i^+\right] \\ E\left[B_i^+\right] &= (1-p_i) E\left[W_i^+\right], \end{aligned} \tag{3.16}$$

since p_i is deterministic. Define

$$\begin{aligned} \mathcal{G}_t &= S_t^2 \\ \mathcal{F}_t &= B_t^2 \\ \mathcal{H}_t &= S_t \cdot B_t. \end{aligned} \tag{3.17}$$

Following similar steps as used to obtain equation (3.9), we can see that

$$\begin{aligned} E\left[\mathcal{G}_{i+1}^-\right] &= E\left[\mathcal{G}_i^+\right]\exp[(2\mu + \sigma_e^2)\Delta t] \\ E\left[\mathcal{F}_{i+1}^-\right] &= E\left[\mathcal{F}_i^+\right]\exp[2r\Delta t] \\ E\left[\mathcal{H}_{i+1}^-\right] &= E\left[\mathcal{H}_i^+\right]\exp[(r+\mu)\Delta t]. \end{aligned} \tag{3.18}$$

Noting that

$$\begin{aligned} \left(W_i^+\right)^2 &= \left(S_i^- + B_i^- + q_i\right)^2 \\ \left(W_i^-\right)^2 &= \left(S_i^- + B_i^-\right)^2, \end{aligned} \tag{3.19}$$

we obtain

$$\begin{aligned} E\left[\left(W_i^+\right)^2\right] &= E\left[\left(W_i^-\right)^2\right] + q_i^2 + 2E\left[S_i^-\right] q_i + 2E\left[B_i^-\right] q_i \\ E\left[\left(W_i^-\right)^2\right] &= E\left[\mathcal{G}_i^-\right] + E\left[\mathcal{F}_i^-\right] + 2E\left[\mathcal{H}_i^-\right]. \end{aligned} \tag{3.20}$$

From equations (3.16), (3.17), and (3.19), we obtain (again noting that p_i is deterministic)

$$\begin{aligned} E\left[\mathcal{G}_i^+\right] &= p_i^2 E\left[\left(W_i^+\right)^2\right] \\ E\left[\mathcal{F}_i^+\right] &= (1-p_i)^2 E\left[\left(W_i^+\right)^2\right] \\ E\left[\mathcal{H}_i^+\right] &= (1-p_i) p_i E\left[\left(W_i^+\right)^2\right]. \end{aligned} \tag{3.21}$$

input: $\{p_0, p_1, \ldots, p_{M-1}\}$ {glide path};
$\{q_0, q_1, \ldots, q_{M-1}\}$ {contributions};
$\{\mu, r, \sigma_e^2, \Delta t\}$ {parameters};
initialize: $E\left[S_0^-\right] = E\left[B_0^-\right] = E\left[\mathcal{G}_0^-\right] = E\left[\mathcal{F}_0^-\right] = E\left[\mathcal{H}_0^-\right] = 0$;

for $i = 0, 1, \ldots, M-1$ **do** {Timestep loop}
$E\left[W_i^+\right] = E\left[S_i^-\right] + E\left[B_i^-\right] + q_i$;
$E\left[\left(W_i^+\right)^2\right] = E\left[\mathcal{G}_i^-\right] + E\left[\mathcal{F}_i^-\right] + q_i^2 + 2E\left[\mathcal{H}_i^-\right] + 2E\left[S_i^-\right] q_i + 2E\left[B_i^-\right] q_i$;
$E\left[S_i^+\right] = p_i E\left[W_i^+\right]$; $E\left[B_i^+\right] = (1-p_i) E\left[W_i^+\right]$;
$E\left[\mathcal{G}_i^+\right] = p_i^2 E\left[\left(W_i^+\right)^2\right]$; $E\left[\mathcal{F}_i^+\right] = (1-p_i)^2 E\left[\left(W_i^+\right)^2\right]$; $E\left[\mathcal{H}_i^+\right] = (1-p_i) p_i E\left[\left(W_i^+\right)^2\right]$;
$E\left[S_{i+1}^-\right] = E\left[S_i^+\right] e^{[\mu \Delta t]}$; $E\left[B_{i+1}^-\right] = E\left[B_i^+\right] e^{[r \Delta t]}$;
$E\left[\mathcal{G}_{i+1}^-\right] = E\left[\mathcal{G}_i^+\right] e^{[(2\mu + \sigma_e^2)\Delta t]}$; $E\left[\mathcal{F}_{i+1}^-\right] = E\left[\mathcal{F}_i^+\right] e^{[2r\Delta t]}$; $E\left[\mathcal{H}_{i+1}^-\right] = E\left[\mathcal{H}_i^+\right] e^{[(r+\mu)\Delta t]}$
end for {End Timestep loop}

{Determine mean and variance at t_M}
$E\left[W_M^-\right] = E\left[S_M^-\right] + E\left[B_M^-\right]$;
$E\left[\left(W_M^-\right)^2\right] = E\left[\mathcal{G}_M^-\right] + E\left[\mathcal{F}_M^-\right] + 2E\left[\mathcal{H}_M^-\right]$;
return mean $= E\left[W_M^-\right]$; variance $= E\left[\left(W_M^-\right)^2\right] - \left(E\left[W_M^-\right]\right)^2$;

ALGORITHM 3.1: *An algorithm for determining the mean and variance of terminal wealth for a given deterministic discrete rebalancing strategy $\{p_0, p_1, \ldots, p_{M-1}\}$ and a schedule of contributions $\{q_0, q_1, \ldots, q_{M-1}\}$, assuming the stochastic processes* (2.4) *and* (2.5).

Given a deterministic glide path $\{p_0, \ldots, p_{M-1}\}$, the mean and variance of terminal wealth can be easily computed using Algorithm 3.1.

For a given specified expected terminal wealth $E\left[W_M^-\right] = d$, the MV optimization problem to determine the optimal (open loop) glide path can be stated as

$$\begin{aligned} &\min_{\{p_0, p_1, \ldots, p_{M-1}\}} \operatorname{Var}\left(W_M^-\right) = E\left[\left(W_M^-\right)^2\right] - d^2 \\ &\text{subject to} \begin{cases} E\left[W_M^-\right] = d \\ E\left[W_M^-\right],\ E\left[\left(W_M^-\right)^2\right] \text{ given by Algorithm 3.1} \\ p_i = p_i(t_i^+);\ 0 \le p_i \le 1 \end{cases} . \end{aligned} \tag{3.22}$$

Note that we impose no-shorting and no-borrowing constraints $0 \le p_i \le 1$, which would be typical in the context of an investor saving for retirement.

Remark 3.1 (Glide Paths in Practice)**.** *It is not by any means clear what criteria are used to construct glide paths for commercial TDFs. We sidestep this problem by using objective function (3.22), which is the optimal deterministic control under MV criteria. In other words, there can be no better glide path, under these criteria.*

3.3 Numerical Solution for the Deterministic Strategy

The objective function for Problem (3.22) can be evaluated very rapidly using Algorithm 3.1, so we can solve for the optimal controls $\{p_0, p_1, \ldots, p_{M-1}\}$ using a numerical optimization technique. We use a Sequential Quadratic Programming (SQP) algorithm (Nocedal and Wright, 2006). Problem (3.22) is not in standard convex optimization form, since the expected value equality constraint is a nonlinear function of the controls p_i. If an SQP algorithm converges, it will converge to a local minimum, and there is no guarantee of convergence to the global minimum. In our numerical tests, we check for possible convergence to local minima by carrying out 10,000 tests, each starting with a different random initial starting guess for the optimal controls $\{p_0, p_1, \ldots, p_{M-1}\}$. In all cases reported here, the SQP algorithm converged to the same solution vector, to within the specified convergence tolerance. This obviously is not a guarantee of convergence to the global minimum, but it is strongly suggestive.

3.4 Deterministic Strategy with Periodic Contributions and Continuous Rebalancing

It is interesting to determine the loss of efficiency in the deterministic case due to discrete rebalancing compared to continuous rebalancing. Of course, in practice trading costs can make high frequency rebalancing very expensive. We consider continuously rebalanced strategies, but with periodic contributions. As in Section 3.2, we specify contributions q_i at times $t_i, i = 0, \ldots, M-1$. There is no contribution at the terminal time $t_M = T$. We assume that the contributions are evenly spaced, so that $t_i - t_{i-1} = \Delta t$. Let $t_i^- = t_i - \epsilon$, $\epsilon \to 0^+$, and $t_i^+ = t_i + \epsilon$. Define the total wealth $W_t = S_t + B_t$, and let $\mathcal{G}_t = W_t^2$. Let

$$W_i^+ = W_{t_i^+};\ W_i^- = W_{t_i^-};\ \mathcal{G}_i^+ = \mathcal{G}_{t_i^+};\ \mathcal{G}_i^- = \mathcal{G}_{t_i^-}. \tag{3.23}$$

At each contribution date t_i we have

$$W_i^+ = W_i^- + q_i;\ \mathcal{G}_i^+ = \mathcal{G}_i^- + 2q_i W_i^- + q_i^2, \tag{3.24}$$

so that

$$E\left[W_i^+\right] = E\left[W_i^-\right] + q_i;\ E\left[\mathcal{G}_i^+\right] = E\left[\mathcal{G}_i^-\right] + 2q_i E\left[W_i^-\right] + q_i^2. \tag{3.25}$$

From the results in Section 3.1, it is easy to see that for a continuously rebalanced deterministic strategy with equity fraction $p(t)$

$$\begin{aligned} E\left[W_{i+1}^-\right] &= E\left[W_i^+\right] e^{(p_i^*(\mu - r) + r)\Delta t} \\ E\left[\mathcal{G}_{i+1}^-\right] &= E\left[\mathcal{G}_i^+\right] \exp\left[2(p_i^*(\mu - r) + r\ )\Delta t + \sigma_e^2 \int_{t_i}^{t_{i+1}} p(s)^2\, ds\right], \end{aligned} \tag{3.26}$$

where

$$p_i^* = (1/\Delta t) \int_{t_i}^{t_{i+1}} p(s)\, ds\ . \tag{3.27}$$

Note that we can consider the continuously rebalanced strategy as the limit of a discretely rebalanced strategy, where we divide the interval between contributions (t_i, t_{i+1}) into sub-timesteps, and let the size of the the sub-timesteps tend to zero. We allow different controls during each sub-timestep. Since the set of admissible controls for the limiting continuously rebalanced strategy is clearly larger than for the discretely rebalanced strategy, the variance of the continuously rebalanced strategy (for a fixed expected value) cannot exceed the variance of the discretely rebalanced strategy.

Before proceeding with our computations, the following result will be useful:

input: $\{p_0^*, p_1^*, \ldots, p_{M-1}^*\}$ {glide path};
$\{q_0, q_1, \ldots, q_{M-1}\}$ {contributions};
$\{\mu, r, \sigma_e^2, \Delta t\}$ {parameters};
initialize: $E\left[W_0^-\right] = E\left[\mathcal{G}_0^-\right] = 0$;

for $i = 0, 1, \ldots, M-1$ **do** {Timestep loop}
$E\left[W_i^+\right] = E\left[W_i^-\right] + q_i$;
$E\left[\mathcal{G}_i^+\right] = E\left[\mathcal{G}_i^-\right] + 2q_i E\left[W_i^-\right] + q_i^2$;
$E\left[W_{i+1}^-\right] = E\left[W_i^+\right] e^{(p_i^*(\mu-r)+r)\Delta t}$;
$E\left[\mathcal{G}_{i+1}^-\right] = E\left[\mathcal{G}_i^+\right] e^{(2\Delta t(p_i^*(\mu-r)+r)+(p_i^*)^2\sigma_e^2\Delta t}$;
end for {End Timestep loop}

{Determine mean and variance at t_M}
return mean $= E\left[W_M^-\right]$; variance $= E\left[\mathcal{G}_M^-\right] - \left(E\left[W_M^-\right]\right)^2$;

ALGORITHM 3.2: *An algorithm for determining the mean and variance of a given deterministic continuously rebalanced strategy $\{p_0^*, p_1^*, \ldots, p_{M-1}^*\}$ and a schedule of contributions $\{q_0, q_1, \ldots, q_{M-1}\}$, assuming the stochastic processes* (2.4) *and* (2.5).

Proposition 3.2 (Optimal strategy: continuously rebalanced, deterministic case)**.** *Consider a market with two assets following the processes* (2.4) *and* (2.5)*, with periodic contributions at discrete times t_i. The MV optimal continuously rebalanced deterministic strategy is to rebalance to a constant equity fraction between contribution times.*

Proof. Consider any strategy $p(t)$. Replace this strategy by the piecewise constant strategy

$$\hat{p}(t) = p_i^*;\ t \in (t_i, t_{i+1}]\ , \tag{3.28}$$

with p_i^* given in equation (3.27). Equations (3.26) now become

$$\begin{aligned} E\left[W_{i+1}^-\right]^* &= E\left[W_i^+\right]^* e^{(p_i^*(\mu-r)+r)\Delta t} \\ E\left[\mathcal{G}_{i+1}^-\right]^* &= E\left[\mathcal{G}_i^+\right]^* \exp\left[2(p_i^*(\mu-r)+r)\Delta t + \sigma_e^2(p_i^*)^2\Delta t\right], \end{aligned} \tag{3.29}$$

where $E[\cdot]^*$ indicates that the strategy (3.28) is used. This new strategy has the same expected value as the original strategy, so that $E\left[W_i^{\pm}\right]^* = E\left[W_i^{\pm}\right], \forall i$. From $Var[W_T] = E[W_T^2]-(E[W_T])^2$, we need only to show that $E\left[\mathcal{G}_M^-\right]^* \leq E\left[\mathcal{G}_M^-\right]$. Assume that $E\left[\mathcal{G}_i^+\right]^* \leq E\left[\mathcal{G}_i^+\right]$. From equations (3.12), (3.26), (3.27), and (3.29), we have $E\left[\mathcal{G}_{i+1}^-\right]^* \leq E\left[\mathcal{G}_{i+1}^-\right]$. From equation (3.25) and the fact that $E\left[W_i^-\right]^* = E\left[W_i^-\right]$, we have $E\left[\mathcal{G}_{i+1}^+\right]^* \leq E\left[\mathcal{G}_{i+1}^+\right]$. Finally, noting that $E\left[\mathcal{G}_0^+\right]^* = E\left[\mathcal{G}_0^+\right]$, the result follows. □

From Proposition 3.2, we can use Algorithm 3.2 to calculate the mean and variance of terminal wealth for a given strategy $\{p_0^*, p_1^*, \ldots, p_{M-1}^*\}$. The optimal continuously rebalanced strategy can be found by using Algorithm 3.2 and solving the optimization problem (3.22), using the methods described in Section 3.3.

4 Adaptive Strategies

To avoid subscript clutter, in the following, we will occasionally use the notation $S_t \equiv S(t), B_t \equiv B(t)$ and $W_t \equiv W(t)$, We now allow the admissible set of controls to depend on the state of the investment portfolio, i.e. $p_i = p\left(S(t_i^+), B(t_i^+), t_i^+)\right) = p\left(S_i^+, B_i^+, t_i^+\right)$, $t_i \in \mathcal{T}$, where $\mathcal{T}$ is the set of rebalancing times. We denote by $X(t) = (S(t), B(t))$, $t \in [0,T]$, the multi-dimensional controlled underlying process, and by $x = (s, b)$ the state of the system.

Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration associated with the wealth process

$$W_t = W(t) = S(t) + B(t) : t \in [0,T] \ .$$

We use $p_i(\cdot)$ to denote the control, representing a strategy as a function of the underlying state, computed at time $t_i^+, t_i \in \mathcal{T}$, i.e. $p_i(\cdot) : (X(t_i^+), t_i^+) \mapsto p_{t_i^+} = p(X(t_i^+), t_i^+)$, for $t_i \in \mathcal{T}$. If we ignore transaction costs, then since we find the optimal strategy amongst all strategies with constant wealth, this is equivalent to $p(X(t_i^+), t_i^+) = p_i\left(W(t_i^+), t_i^+\right)$.

Let $\mathcal{Z}$ represent the set of admissible values of the control $p_i(\cdot)$. An admissible control $\mathcal{P} \in \mathcal{A}$, where $\mathcal{A}$ is the admissible control set, can be written as

$$\mathcal{P} = \{p_i \in \mathcal{Z} \ : \ i = 0, \ldots, M-1\} \tag{4.1}$$

We also define $\mathcal{P}_n \equiv \mathcal{P}_{t_n^+} \subset \mathcal{P}$ as the tail of the set of controls in $[t_n, t_{n+1}, \ldots, t_{M-1}]$, i.e.

$$\mathcal{P}_n = \{p_n, \ldots, p_{M-1}\} \tag{4.2}$$

4.1 Time Consistent MV

With these notational conventions, for a given scalarization parameter $\rho > 0$ and an intervention time t_n, we define the scalarized time consistent MV problem $(TCMV_{t_n}(\rho))$ and the value function $V(s,b,t)$: as follows:

$$\begin{aligned}
(TCMV_{t_n}(\rho)): \qquad V\left(s,b,t_n^-\right) &= \sup_{\mathcal{P}_n \in \mathcal{A}} \Bigg\{ E_{\mathcal{P}_n}^{X_n^+, t_n^+}\left[W_T\right] - \\
&\qquad \rho(W_n^+) Var_{\mathcal{P}_n}^{X_n^+, t_n^+}\left[W_T\right] \Bigg| X(t_n^-) = (s,b) \Bigg\} && (4.3) \\
\text{s.t. } \mathcal{P}_n &= \left\{p_n, \mathcal{P}_{n+1}^*\right\} := \left\{p_n, p_{n+1}^*, \ldots, p_{M-1}^*\right\} && (4.4) \\
&\quad \text{where } \mathcal{P}_{n+1}^* \text{ is optimal for problem } \left(TCMV_{t_{n+1}}(\rho)\right)
\end{aligned}$$

$$\text{subject to} \quad \begin{cases} (S_t, B_t) \text{ follow processes (2.4)-(2.5)}; \quad t \notin \mathcal{T} \\ W_n^+ = s + b + q_n \, ; \ X_n^+ = (S_n^+, B_n^+) \\ S_n^+ = p_n W_n^+ \, ; \ B_n^+ = W_n^+ - S_n^+ \\ p_\ell \in \mathcal{Z} = [0,1] \, ; \ \ell = n, \ldots, M-1 \end{cases} \tag{4.5}$$

Remark 4.1 (Time Consistent Constraint)**.** *Time consistency is enforced via the constraint (4.4). If this constraint is eliminated, we would obtain the pre-commitment MV solution (Basak and Chabakauri, 2010).*

The definition of time consistency with $\rho = const.$ was originally suggested in Basak and Chabakauri (2010). We also consider the case of wealth-dependent parameter ρ suggested in Björk et al. (2014).

Björk et al. (2014) note that the time consistent strategy for multi-period MV optimality (with no constraints) has the property that the amount invested in the risky asset is deterministic (i.e. not a function of W_t). Björk et al. argue that this is economically unreasonable, and suggest using a wealth-dependent risk aversion parameter to ameliorate this difficulty. However, Wang and Forsyth

(2011) show that using a wealth-dependent risk aversion has strange effects. In particular, adding constraints to the strategy results in an efficient frontier which plots *higher* than the unconstrained efficient frontier.

In this work, the cases considered for time consistent MV strategies are as follows:

$$\rho(W_t,t) \quad = \quad \begin{cases} \hat{\rho} & \text{Case 1} \\ \frac{\hat{\rho}}{W_t} & \text{Case 2} \end{cases}$$
$$\hat{\rho} = const. > 0 \ . \tag{4.6}$$

The time consistent constraint (4.4) for the discrete rebalancing case was also considered in Bjork and Murgoci (2014); Staden et al. (2018); Landriault et al. (2018).

Remark 4.2 (Numerical Solution: Time Consistent MV)**.** *Due to the constraints on the control (equation (4.5)), a closed form solution is not possible. We formulate problem* $(TCMV_{t_n}(\rho))$ *as a dynamic programming problem, which requires solution of a system of nonlinear Hamilton-Jacobi-Bellman (HJB) Partial Integro Differential Equations (PIDEs), and use the techniques discussed in Wang and Forsyth (2011); Staden et al. (2018). In particular, we use the* ϵ*-monotone Fourier method discussed in Forsyth and Labahn (2018) to solve the PIDEs between rebalancing dates.*

Remark 4.3 (Specifying $E^{X_0^+,t_0^+}[W_T]$)**.** *In order to provide a fair comparison amongst different strategies, we will determine the parameter* $\hat{\rho}$ *so that* $E^{X_0^+,t_0^+}[W_T] = d$*, where* d *is fixed. To be precise, let* $\mathcal{P}(\hat{\rho})$ *be the optimal control for problem* $(TCMV_{t_0}(\hat{\rho}))$*. Then, we determine the value of* $\hat{\rho}^*$ *such that*

$$f(\hat{\rho}^*) = E^{X_0^+,t_0^+}_{\mathcal{P}(\hat{\rho}^*)}(W_T) - d = 0 \ . \tag{4.7}$$

We solve equation (4.7) by a Newton iteration. Each evaluation of $f(\hat{\rho}^*)$ *requires solving a system of PIDEs. This can be done efficiently by determining an approximate value for* $\hat{\rho}^*$ *on a coarse grid, and then using this as the initial estimate for the Newton iteration on a sequence of grids.*

4.2 Expected Quadratic Shortfall

In the insurance literature, as noted by Vigna (2014) and Menoncin and Vigna (2017), a common investment objective function in the DC plan context is based on minimizing the expected quadratic shortfall with respect to a fixed target (real) terminal wealth W^*. We define the time consistent quadratic shortfall problem $(TCQS_{t_n}(W^*))$ and value function $V(s,b,t)$ as:

$$(TCQS_{t_n}(W^*)): \qquad V\left(s,b,t_n^-\right) \quad = \quad \inf_{\mathcal{P}_n \in \mathcal{A}} \left\{ E^{X_n^+,t_n^+}_{\mathcal{P}_n}\left[\min(W_T - W^*,0)^2\right] \middle| X(t_n^-) = (s,b) \right\} \tag{4.8}$$

$$\text{s.t.} \qquad \text{Surplus cash invested in the bond} \tag{4.9}$$

$$\text{subject to} \quad \begin{cases} (S_t, B_t) \text{ follow processes (2.4)-(2.5)}; \quad t \notin \mathcal{T} \\ W_n^+ = s + b + q_n \ ; \ X_n^+ = (S_n^+, B_n^+) \\ S_n^+ = p_n W_n^+ \ ; \ B_n^+ = W_n^+ - S_n^+ \\ p_\ell \in \mathcal{Z} = [0,1] \ ; \ \ell = n, \ldots, M-1 \end{cases} \tag{4.10}$$

Remark 4.4 (Dynamic Programming Solution)**.** *Since problem* $(TCQS_{t_n}(W^*))$ *is a simple expectation, it can be formulated as a dynamic programming problem, and hence is trivially time consistent for fixed target* W^**. There is no need to enforce a time consistent constraint in this case.*

Note that problem $(TCQS_{t_n}(W^*))$ penalizes shortfall with respect to W^*. It is assumed that the sole concern of the DC plan investor is with a shortfall with respect to the target wealth W^*. The investor is assumed to be indifferent to any terminal wealth $W_T > W^*$. This implies that we take the smallest risk possible to realize a terminal wealth close to W^*. Vigna (2014) and Menoncin and Vigna (2017) suggest that this is a reasonable strategy for a DC plan investor.

In fact, Vigna (2014) shows that for the case where the risky asset follows Geometric Brownian Motion and rebalancing is continuous, the optimally controlled wealth never exceeds W^* at any time. However, in our case (discrete rebalancing and jumps) this is no longer true. Constraint (4.9) is required in this case to fully specify the problem. Let

$$Q_\ell = \sum_{j=\ell+1}^{j=M-1} e^{-r(t_j - t_\ell)} q_j \tag{4.11}$$

be the discounted future contributions as of time t_ℓ. If

$$W_i^+ > W^* e^{-r(T-t_i)} - Q_i, \tag{4.12}$$

then the optimal strategy is to (i) invest $W^* e^{-r(T-t_i)} - Q_i$ in the risk-free bond and (ii) invest cash $c_i = W_i^- + q_i - \big(W^* e^{-r(T-t_i)} - Q_i\big)$ with an arbitrary allocation to the stock index and bonds. This is optimal in this case since $E^{X_i^+, t_i^+}\big[\min(W^* - W_T, 0)^2\big] = 0$, which is the minimum of Problem (4.8). Consistent with the terminology in the literature (Bauerle and Grether, 2015), we term the amount

$$c_i \;=\; \max\Bigg(W_i^- + q_i - \Big(W^* e^{-r(T-t_i)} - Q_i\Big), 0\Bigg) \tag{4.13}$$

as *surplus cash*. We can invest the surplus cash in the stock and bond in any proportion. In our case, we impose the constraint that the surplus cash is invested in the bond.

Remark 4.5 (Numerical Solution: Quadratic Shortfall)**.** *The constrained problem $(TCQS_{t_n}(W^*))$ has no closed form solution in general. We formulate problem $(TCQS_{t_n}(W^*))$ as a dynamic program, which requires solution of nonlinear Hamilton-Jacobi-Bellman PIDEs, which we solve using the techniques in Forsyth and Labahn (2018). In order to fix $E^{X_0^+, t_0^+}_{\mathcal{P}(W^*)}[W_T] = d$, we solve for the value of W^* such that*

$$f(W^*) = E^{X_0^+, t_0^+}_{\mathcal{P}(W^*)}[W_T] - d = 0 \tag{4.14}$$

using a Newton iteration, as in Remark 4.3.

4.3 Relation to Pre-commitment MV

The pre-commitment MV problem $(PCMV_{t_n}(\lambda))$ for a fixed constant $\lambda > 0$, and the value function $V(s,b,t)$ can be formally defined as:

$$
\begin{aligned}
(PCMV_{t_n}(\lambda)): \qquad V\left(s,b,t_n^-\right) &= \sup_{\mathcal{P}_n \in \mathcal{A}} \Bigg\{ E_{\mathcal{P}_n}^{X_n^+,t_n^+}[W_T] - \\
&\qquad \lambda\, Var_{\mathcal{P}_n}^{X_n^+,t_n^+}[W_T] \,\Bigg|\, X(t_n^-) = (s,b) \Bigg\} && (4.15)\\
\text{s.t.} \qquad & \text{Surplus cash withdrawn} && (4.16)\\
\text{subject to} \qquad & \begin{cases} (S_t, B_t) \text{ follow processes (2.4)-(2.5)}; \quad t \notin \mathcal{T} \\ W_n^+ = s + b + q_n \,;\; X_n^+ = (S_n^+, B_n^+) \\ S_n^+ = p_n W_n^+ \,;\; B_n^+ = W_n^+ - S_n^+ \\ p_\ell \in \mathcal{Z} = [0,1] \;;\; \ell = n, \ldots, M-1 \end{cases} && (4.17)
\end{aligned}
$$

Note that we have dropped the time consistent constraint from Problem $(PCMV_{t_n}(\lambda))$, but included the constraint that surplus cash is withdrawn. Let $\mathcal{P}_{t_n}^* = \{p_n^*, p_{n+1}^*, \ldots, p_{M-1}^*\}$ be the optimal controls for Problem $(PCMV_{t_n}(\lambda))$ and let $\mathcal{P}_{t_\ell}^*$ be the optimal controls for Problem $(PCMV_{t_\ell}(\lambda))$ with $t_\ell > t_n$. Then, the pre-commitment strategy can be time inconsistent in the sense that in general

$$
p_k^* \in \mathcal{P}_{t_\ell}^* \quad \neq \quad p_k^* \in \mathcal{P}_{t_n}^* \;;\; k > \ell > n \tag{4.18}
$$

However, is interesting to note the following result proven in (Li and Ng, 2000; Zhou and Li, 2000):

Theorem 4.1 (Embedding Result)**.** *Let $\mathcal{P}_{t_n}^{pcmv}(\lambda)$ be the optimal control for problem $(PCMV_{t_n}(\lambda))$. Let $\mathcal{P}_{t_n}^{tcqs}(W^*)$ be the optimal control for problem $(TCQS_{t_n}(W^*))$. Suppose we fix λ for problem $(PCMV_{t_n}(\lambda))$. Then there exists a $W_{t_n}^*(\lambda)$ such that $\mathcal{P}_{t_n}^{tcqs}(W_{t_n}^*(\lambda)) = \mathcal{P}_{t_n}^{pcmv}(\lambda)$, assuming surplus cash (defined in equation (4.13)) is withdrawn for both problems.*

Remark 4.6 (Extension to surplus cash withdrawal)**.** *The optimality of withdrawing surplus cash in the pre-commitment case is discussed in Cui et al. (2012), Bauerle and Grether (2015), and Dang and Forsyth (2016).*

At first sight, it is difficult to reconcile this result with the fact that problem $(TCQS_{t_n}(W_{t_n}^*(\lambda)))$ is obviously time consistent, while problem $(PCMV_{t_n}(\lambda))$ is not time consistent. However, it turns out that the equivalent target in Theorem 4.1 for fixed λ is such that in general, $W_{t_n}^*(\lambda) \neq W_{t_\ell}^*(\lambda),\ \ell \neq n$. As noted by Cong and Oosterlee (2016a;b), the pre-commitment MV strategy is consistent with a fixed target, but not with a risk aversion attitude. Conversely, the time consistent MV strategy has a constant risk aversion, but is not consistent with a fixed investment target. Menoncin and Vigna (2017) and Vigna (2017) provide further insight into this issue.

Other definitions of time consistent MV are also possible. For example, He and Jiang (2017) suggest using an expected value constraint, at each instant in time, based on current wealth and a desired growth rate. Note that this objective does not attempt to hit a fixed target. Instead, the target simply adjusts to current wealth. We will not pursue these ideas further in this work: all of the adaptive strategies that we consider are formally time consistent.

5 Data and Parameters

The parameters of equations (2.4) and (2.5) are estimated using data from the Center for Research in Security Prices (CRSP) on a monthly basis over the 1926-2015 period.[2] Our base case tests use the CRSP 3-month Treasury bill (T-bill) index for the risk-free asset and the CRSP value-weighted total return index for the risky asset. This latter index includes all distributions for all domestic stocks trading on major U.S. exchanges. As an alternative case for additional illustrations, we replace the above two indexes by a 10-year Treasury index and the CRSP equal-weighted total return index.[3] All of these various indexes are in nominal terms, so we adjust them for inflation by using the U.S. CPI index, also supplied by CRSP. We use real indexes since investors saving for retirement should be focused on real (not nominal) wealth goals.

Appendix A discusses the methods used to calibrate the model parameters to the historical data. We use both a threshold technique (Cont and Mancini, 2011) and maximum likelihood (ML) estimation. The threshold estimator requires a parameter α, described in Appendix A. Briefly, we identify a jump if the magnitude of the observed return in a month is greater than α standard deviations from the mean expected return assuming geometric Brownian motion. Given our data frequency, setting $\alpha = 3$ is a sensible choice (Forsyth and Vetzal, 2017). Annualized estimated parameters using both the threshold method with $\alpha = 3$ and ML for both the value-weighted and equal-weighted indexes are provided in Table 5.1. As might be expected due to the small firm effect, the equal-weighted index has slightly higher estimated diffusion parameters (μ and σ). It also has a higher estimated probability of an upward jump, and jumps that tend to be a little larger in magnitude. More importantly for our purposes, the ML parameter estimates imply much more frequent and smaller jumps on average for both indexes. From the perspective of a long-term investor, it is probably more appropriate to model infrequent larger jumps. Hence we have a preference for the threshold estimates, so we use them in the numerical examples below. We also note that Dang et al. (2017) and Forsyth and Vetzal (2017) conduct some tests using both ML and threshold techniques. A range of values for α are used to estimate the jump diffusion parameters. As one example, Forsyth and Vetzal (2017) compute the optimal adaptive strategy using ML estimates, and then apply this control in a synthetic market where the stochastic process follows parameters which are estimated by thresholding. The investment results are robust to this form of parameter misspecification.

Table 5.2 shows the average annualized returns and volatilities for the real 3-month T-bill and 10-year U.S. Treasury indexes over the entire sample period from 1926 to 2015. The 10-year index earned an average return of about 130 basis points per year over the 3-month index during this time. The volatility of the long-term index was more than three times higher than that of the short-term index, but still relatively small in comparison to the volatility of the equity market index from Table 5.1.[4]

[2]More specifically, results presented here were calculated based on data from Historical Indexes, ©2015 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

[3]The 10-year Treasury index was constructed from monthly returns from CRSP back to 1941. The data for 1926-1941 were interpolated from annual returns in Homer and Sylla (2005).

[4]Note that the effective volatility of the equity market index reflects diffusive volatility σ as well as contributions to volatility from jumps. The effective volatility is $\sigma_{eff} = \sigma^2 + \lambda E[(\xi - 1)^2]$.

Method	μ	σ	λ	p_{up}	η_1	η_2
Real CRSP Value-Weighted Index						
ML	.08326	.12611	3.0881	0.09963	10.837	18.913
threshold ($\alpha = 3$)	.08889	.14771	.32222	0.27586	4.4273	5.2613
Real CRSP Equal-Weighted Index						
ML	.10735	.14256	2.8166	.14407	8.3486	14.963
threshold ($\alpha = 3$)	.11833	.16633	.40000	.33334	3.6912	4.5409

Table 5.1: *Estimated annualized parameters for double exponential jump diffusion model. Value-weighted and equal-weighted CRSP indexes, deflated by the CPI. Sample period 1926:1 to 2015:12. "ML" refers to Maximum Likelihood estimation.*

	Real 3-month T-bill Index	Real 10-year Treasury Index
Mean return	.00827	.02160
Volatility	.019	.065

Table 5.2: *Mean annualized real rates of return for bond indexes ($log[B(T)/B(0)]/T$). Volatilities (annualized) computed using log returns. Sample period 1926:1 to 2015:12.*

6 Numerical Examples

6.1 Overview

We consider the input data summarized in Table 6.1. An investor with a horizon of 30 years makes real contributions each year of $10, allocated between the stock index and the bond index.

6.2 Deterministic Strategies: Discrete vs. Continuous Rebalancing

We first compare the use of continuous and discrete rebalancing, for the deterministic strategies, with periodic contributions. Table 6.2 compares the MV optimal results for the deterministic strategies for both discretely and continuously rebalanced cases. In each case, $E[W_T]$ is set equal to that for a discretely rebalanced constant proportion strategy, with $p = .5$. The constant proportion

	Base Case	Alternative Case
Investment horizon (years)	30	30
Equity market index	Value-weighted	Equal-weighted
Risk-free asset index	3-month T-bill	10-year Treasury
Initial investment W_0 ($)	0.0	0.0
Real investment each year ($)	10.0	10.0
Rebalancing interval (years)	1	1

Table 6.1: *Input data for examples. Cash is invested at $t = 0, 1, \ldots, 29$ years. Market parameters are provided in Tables 5.1 and 5.2.*

	Base Case		Alternative Case	
Strategy	$E[W_T]$	$std[W_T]$	$E[W_T]$	$std[W_T]$
Optimal deterministic (discrete)	705.6	340.6	1085.2	846
Constant proportion (continuous)	705.6	337.6	1085.2	814
Optimal deterministic (continuous)	705.6	329.5	1085.2	802

TABLE 6.2: *Comparison of discretely and continuously rebalanced strategies for input data given in Table 6.1 and corresponding parameters from Table 5.1 (threshold) and 5.2.*

(continuously rebalanced) weights which generate these expected values of terminal wealth are $p = .510$ (base case) and $p = .512$ (alternative case). As expected, the continuously rebalanced strategy is superior to the discretely rebalanced policy, but not by much. This has the practical implication that infrequent rebalancing does not reduce efficiency to a large degree, while reducing transaction costs. Based on these results, we will assume discrete rebalancing in the following.

6.3 Base Case: CRSP Value-Weighted Index and 3-month T-bill Index

We next focus attention on the base case input data summarized in Table 6.1. An investor with a horizon of 30 years makes real contributions each year of \$10, allocated between the CRSP value-weighted and 3-month T-bill indexes and rebalanced annually.

6.3.1 Synthetic Market - Base Case

We refer to a market where the underlying stock and bond indexes follow processes (2.4) and (2.5), with fixed parameters given in Tables 5.1 and 5.2, as a *synthetic market*. In other words, this is a market based on the historical (constant) estimated parameters. We are careful to distinguish tests in a synthetic market with tests that use actual historical returns (*bootstrap resampling*), as discussed below in Section 6.3.2.

We first use a constant proportion strategy ($p = 0.5$) and determine the expected value of the terminal real wealth for this strategy. We then use this expected value as a constraint and determine the optimal deterministic strategy, which is the solution of problem (3.22). We use the same expected value as a constraint and solve for the optimal adaptive strategies: time consistent MV (Section 4.1) and quadratic shortfall (Section 4.2).

We evaluate the performance of the various strategies using Monte Carlo simulation in the synthetic market. This case constitutes the best possible context for both the optimal deterministic and the optimal adaptive strategies since the associated control parameters are based on perfect knowledge of the stochastic properties of the market.

We will report various statistics of the final wealth W_T for these strategies. We report the mean, median, probability of shortfall, and Conditional Value at Risk at the 5% level, which we denote by CVAR(5%). In the case of the quadratic shortfall strategy, we include the surplus cash (4.13) in all statistics except for the standard deviation. Along any path where surplus cash (as defined in equation (4.13)) is generated, the risk of shortfall is identically zero. However, including surplus cash will generally increase the standard deviation. Hence it seems non-informative in this case to include the surplus cash in this measure of risk.

Let $f(W_T)$ be the probability density of the final wealth distribution. We define CVAR(α) as

$$\text{CVAR}(\alpha) \quad = \quad \frac{\int_{-\infty}^{u^*} u \ f(u) \ du}{\alpha} \ , \tag{6.1}$$

where u^* is defined by

$$\int_{\infty}^{u^*} f(u) \ du = \alpha \ . \tag{6.2}$$

CVAR(α) has the convenient interpretation as the mean of the worst α fraction of outcomes. Note that we define CVAR(α) in terms of final wealth, not losses. Hence, larger CVAR(α) corresponds to smaller risk.

We can get an idea of the accuracy of our numerical results by examining Table 6.3. We first compute the optimal control by solving the HJB equation using a set of increasing grid sizes. As a by-product of computing the strategy, we also compute an approximation to $E[W_T]$ and $std[W_T]$. We store the control determined from the HJB equation solve, and then carry our Monte Carlo simulations, using varying number of simulations, N_{sim}. Note that the Monte Carlo method used exact timestepping between rebalancing dates. We can see that the Monte Carlo and PIDE computed values are in good agreement. In the following, for the synthetic market results, we will report $E[W_T]$ and $std[W_T]$ from the PIDE computations, using the 1024×609 grid. The other statistics will be determined by using Monte Carlo simulations (but using the HJB computed control) using $N_{sim} = 160{,}000$.

The computational cost for computing the optimal strategy is fairly modest. As an example, computing the quadratic shortfall strategy with a specified $E[W_T]$ requires 5 Newton iterations on the coarse 512×305 grid, followed by 2 Newton iterations on the fine 1024×609 grid. Each Newton iteration requires two HJB PIDE solves. The total CPU time for all of the Newton iterations was about 45 seconds on a standard desktop.

Table 6.4 compares the results for the constant proportion, optimal deterministic, and optimal adaptive strategies. By design, all three strategies have the same expected real terminal wealth. The optimal deterministic standard deviation is about 0.98 times that of the constant proportion strategy, so the optimal deterministic strategy offers little improvement over a simpler constant proportion strategy with the same expected terminal wealth.

Amongst the adaptive strategies, the performance of the time consistent MV (Case 2) strategy is quite poor. In contrast, the time consistent MV (Case 1) strategy outperforms the deterministic strategies by almost all risk measures. Quadratic shortfall is superior to the time consistent MV (Case 1) policy by all measures except CVAR (5%).

Recall that Proposition 3.1 shows that a constant proportion strategy dominates any deterministic glide path by MV criteria, assuming that the portfolio is continuously rebalanced and that there is a lump sum initial investment. That result clearly does not hold in current context with annual rebalancing and contributions. However, the results from Table 6.4 are not very encouraging for the optimal deterministic strategy as it gives just very slight improvement over the simpler constant proportion alternative. Moreover, this is in a context that is tailor made for the deterministic strategy because the market simulations here use parameters and stochastic processes that exactly match those assumed when determining the optimal controls.

The intuition underlying the marginal improvement of the optimal deterministic strategy compared to the constant proportion strategy is as follows. As the time in the strategy becomes large, the marginal amount contributed is small compared to the accumulated wealth (on average), hence the optimal strategy tends to a constant proportion (i.e. this begins to resemble the lump sum case,

HJB Equation			Monte Carlo			
Grid	$E[W_T]$	$std[W_T]$	$E[W_T]$	$std[W_T]$	CVAR (5%)	N_{sim}
512×305	705.6	153.1	704.9 (.99)	153.5	237.4	160,000
			705.5 (.49)	153.0	238.1	640,000
1024×609	705.6	152.9	704.9 (.99)	153.4	237.4	160,000
			705.4 (.49)	153.0	238.0	640,000
2048×1217	705.6	152.8	704.9 (.99)	153.4	237.3	160,000
			705.5 (.49)	153.0	237.9	640,000

TABLE 6.3: *Synthetic market results, for the Quadratic Shortfall strategy, tests of accuracy. Grid refers to the grid used to solve the HJB PDE: $n_x \times n_b$, where n_x is the number of nodes in the* $\log S$ *direction, and n_b is the number of nodes in the B direction. N_{sim} is the number of Monte Carlo simulations. The numbers in brackets are the standard errors at the 99% confidence level. Base case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2.*

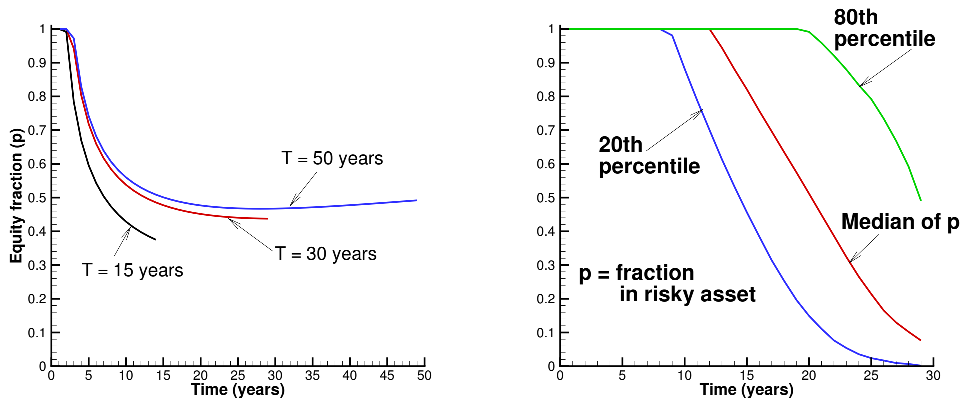
					Probability of Shortfall	
Strategy	$E[W_T]$	$Median[W_T]$	$std[W_T]$	CVAR (5%)	$W_T < 500$	$W_T < 600$
Constant proportion ($p = 0.5$)	705.6	628	349	291	.28	.45
Optimal deterministic	705.6	630	341	306	.27	.45
Time consistent MV (Case 1)	705.6	688	224	305	.17	.33
Time consistent MV (Case 2)	705.6	592	477	177	.38	.51
Quadratic Shortfall	705.6	776	153	237	.12	.17

TABLE 6.4: *Synthetic market results from 160,000 Monte Carlo simulation runs for base case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2.*

and we know from Proposition 3.1 that a constant proportion strategy will be superior to any glide path in this case).

Figures 6.1 and 6.2 show the optimal controls for both the deterministic and adaptive strategies. As a comparison, we show the deterministic control for $T = 15, 30, 50$ years in Figure 6.1(a). In each case, $E[W_T]$ is set to the expected final wealth for the constant proportion $p = 0.5$ case. Note that $p(t) \rightarrow 0.5$ as (T, t) increase, consistent with the intuition given above. In the adaptive cases, the control is a function of the current wealth. For ease of illustration, we show the median and the 20th and 80th percentiles of $p(W_t, t)$ for the quadratic shortfall case with $T = 30$ years in Figure 6.1(b), which we compute by Monte Carlo simulation. Although the median value of p corresponds in a general way to the standard glide path (starting with a high equity allocation and declining as the investment horizon is approached), the wide range of values between the two percentiles shown for values of $t > 10$ years shows that the quadratic shortfall strategy depends significantly on accumulated wealth.

Figure 6.2 compares the controls for the time consistent MV (Case 1) and time consistent MV (Case 2) strategies (see equation (4.6)). The poor performance of the Case 2 strategy (wealth dependent risk aversion parameter) can be traced to the rather bizarre controls generated. The portfolio is essentially all bonds for the first 15 years, followed by a rapid transition to all equities. This result has also been observed in Staden et al. (2018), where this effect is explained on the basis of asymptotic analysis. Intuitively, Case 2 (see equation (4.6)) has a very large effective risk



(a) *Optimal deterministic control for various investment horizons T.*

(b) *Quadratic shortfall adaptive control for the case with $T = 30$ years.*

FIGURE 6.1: *Properties of optimal strategies using base case input data from Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Synthetic market. In each case, $E[W_T]$ is constrained to match that of a constant proportion strategy with $p = 0.5$. Figure 6.1(b) is based on 160,000 Monte Carlo simulation runs. Quadratic shortfall is discussed in Section 4.2.*

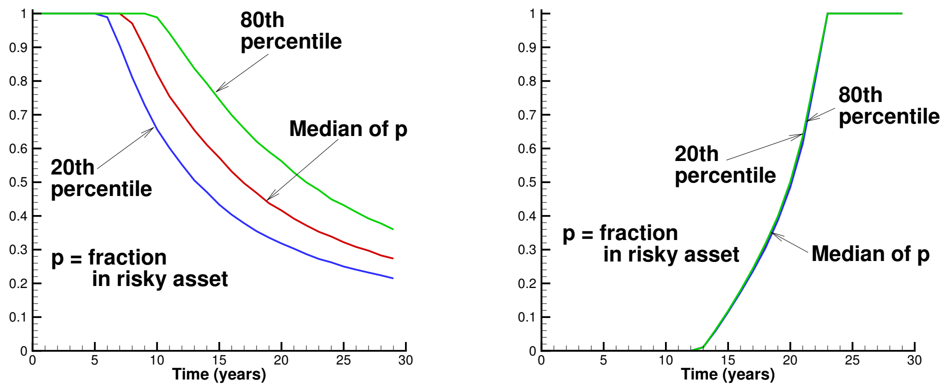
aversion parameter when W_t is small, which is the case at early times. Hence the strategy is to invest the entire portfolio in bonds. At later times, after the wealth has increased (mainly due to contributions) the effective risk aversion parameter has decreased enough to permit investment in risky assets. However, by this time, in order to hit the specified expected value of the terminal wealth, the equity fraction is increased to the largest possible value. For a more rigorous analysis of this effect, we refer the reader to Staden et al. (2018). Further insight into the various adaptive strategies can be obtained by examining the strategy heat maps in Figure 6.3.

6.3.2 Resampled Historical Data - Base Case

Although it is useful to examine strategies for synthetic markets with parameters obtained from historical data, it is perhaps more convincing to see how the various strategies would have performed on actual historical data. We use bootstrap resampling to study this.

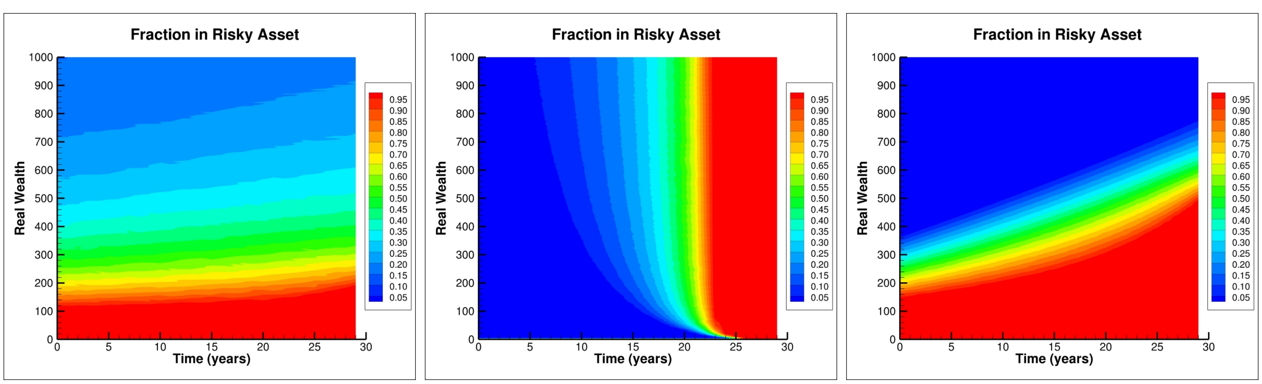
A single bootstrap resampled path is constructed as follows. Suppose the investment horizon is T years. We divide this total time into k blocks of size b years, so that $T = kb$. We then select k blocks at random (with replacement) from the historical data (from both the deflated stock and bond indexes). Each block starts at a random month. We then form a single path by concatenating these blocks. Since we sample with replacement, the blocks can overlap. To avoid end effects, the historical data is wrapped around, as in the circular block bootstrap (Politis and White, 2004; Patton et al., 2009). We repeat this procedure for many paths. The sampling is done in blocks in order to account for possible serial dependence effects in the historical time series. The choice of blocksize is crucial and can have a large impact on the results (Cogneau and Zakalmouline, 2013). We simultaneously sample the real stock and bond returns from the historical data. This introduces random real interest rates in our samples, in contrast to the constant interest rates assumed in the synthetic market tests and in the determination of the optimal controls.

To reduce the impact of a fixed blocksize and to mitigate the edge effects at each block end, we use the stationary block bootstrap (Politis and White, 2004; Patton et al., 2009). The blocksize is randomly sampled from a geometric distribution with an expected blocksize $\hat{b}$. A precise pseudo-



(a) *Time consistent MV: Case 1.* (b) *Time consistent MV: Case 2.*

FIGURE 6.2: *Time consistent MV controls (see Section 4.1) using base case input data from Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Synthetic market. $E[W_T]$ is constrained to match that of a constant proportion strategy with $p = 0.5$. 160,000 Monte Carlo simulation runs.*



(a) *TCMV: Case 1.* (b) *TCMV: Case 2.* (c) *Quadratic shortfall*

FIGURE 6.3: *Heat maps of controls using base case input data from Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Synthetic market. $E[W_T]$ is constrained to match that of a constant proportion strategy with $p = 0.5$. TCMV: time consistent MV, Section 4.1.*

Data series	Optimal expected block size $\hat{b}$ (months)
Real 3-month T-bill index	50.1
Real 10-year Treasury index	4.7
Real CRSP value-weighted index	1.8
Real CRSP equal-weighted index	10.4

TABLE 6.5: *Optimal expected blocksize $\hat{b} = 1/v$ when the blocksize follows a geometric distribution $Pr(b = k) = (1 - v)^{k-1}v$. The algorithm in Patton et al. (2009) is used to determine $\hat{b}$.*

code description of the block bootstrap sampling technique is given in Appendix B.

The optimal choice for $\hat{b}$ is determined using the algorithm described in Patton et al. (2009). This approach has also been used in other tests of portfolio allocation problems recently (e.g. Dichtl et al., 2016). Calculated optimal values for $\hat{b}$ for the various indexes are given in Table 6.5.

We compute and store the optimal strategies (deterministic and adaptive) for the base case input data from Table 6.1 and the corresponding market parameters from Tables 5.1 (threshold) and 5.2. All strategies are constrained to have $E[W_T] = 705.6$ (in the synthetic market). We then apply these strategies using bootstrap resampling, based on the historical monthly data from 1926:1-2015:12. Of course, the resampled means will not be precisely the same and equal to 705.6 for this test.

To get some idea of the effect of the number of samples used in the resampling simulations, we report results in Table 6.6 for a blocksize of two years with increasing numbers of resamples. In the following, we will report results using 10,000 bootstrap resamples.

The results for various blocksizes are shown in Table 6.7. We omit the results for time consistent MV (Case 2) due to their poor performance. Choosing a blocksize that is too large will result in artificially low standard deviations. Table 6.7 indicates that the results are not too sensitive to expected blocksizes in the range of 0.5 to 2 years. Generally, the results in Table 6.7 are quite comparable to those from the synthetic market reported in Table 6.4. Quadratic shortfall has the best statistics except for CVAR. Time consistent MV (Case 1) outperforms the deterministic strategies by all measures. In summary, quadratic shortfall is superior to the other strategies over a wide range of outcomes, except in the extreme left tail.

Figure 6.4 shows the cumulative distribution functions for the various strategies computed using bootstrap resampling of the actual historical data. Again, the cumulative distribution function for the optimal deterministic strategy is very close to that for the constant proportion strategy. The adaptive strategies clearly reduce risk over a wide range of outcomes, but at the cost of reducing the probability of very large gains. We suggest that this is an appropriate trade off for retirement savings. Looking at Figure 6.4, we can see that the left tail risk of the quadratic shortfall strategy is slightly worse than the left tail risk for the time consistent MV (Case 1) strategy. This is, of course, consistent with the CVAR results in Table 6.7.

Note that the median terminal wealth for the quadratic shortfall policy is significantly larger than for the other strategies, for all blocksizes. The probabilities of shortfall (for moderate values of the terminal wealth) are also much smaller than for the other strategies, for all blocksizes.

6.4 Alternative Case: CRSP Equal-Weighted Index and 10-year Treasury Index

As a check on the robustness of our results, we use alternative assets. In particular, as indicated in Table 6.1, we replace the CRSP value-weighted index with its equal-weighted counterpart, and we substitute the 10-year Treasury bond index for the 3-month Treasury bill index. See Tables 5.1

$E[W_T]$	$Median[W_T]$	$std[W_T]$	CVAR (5%)	Number of resamples
699.2	756.8	137.2	279.1	1×10^4
699.6	758.7	138.1	275.2	1×10^5
699.4	758.6	138.1	272.7	1×10^6

TABLE 6.6: *Stationary block bootstrap: effect of number of bootstrap resamples, for base case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Quadratic shortfall strategy, expected blocksize: 2 years. Rebalanced annually, based on the synthetic market with* $E[W_T] = 705.6$. *Calculations based on bootstrap resamples of historical data for the period 1926:1 to 2015:12.*

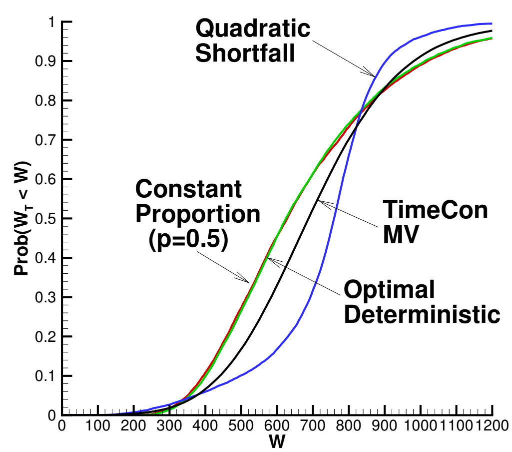


FIGURE 6.4: *Cumulative distribution functions using base case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2.* "TimeCon MV" *refers to time consistent MV, Case 1, as in equation (4.6). Distributions are computed using 10,000 bootstrap resamples of the historical data from 1926:1 to 2015:12. Expected blocksize* $\hat{b} = 2$ *years. Strategies are based on the synthetic market with* $E[W_T] = 705.6$ *in all cases.*

Strategy	$E[W_T]$	$Median[W_T]$	$std[W_T]$	CVAR (5%)	Probability of Shortfall $W_T < 500$	Probability of Shortfall $W_T < 600$
			Expected Blocksize $\hat{b} = 0.25$ years			
Constant proportion ($p = .5$)	677	621	276	298	.27	.46
Optimal deterministic	676	623	268	309	.27	.46
Time consistent MV (Case 1)	684	676	195	313	.17	.34
Quadratic shortfall	698	761	146	256	.11	.17
			Expected Blocksize $\hat{b} = 0.5$ years			
Constant proportion ($p = .5$)	680	627	278	295	.28	.46
Optimal deterministic	679	624	272	306	.28	.46
Time consistent MV (Case 1)	686	676	197	313	.17	.34
Quadratic shortfall	695	758	147	252	.12	.18
			Expected Blocksize $\hat{b} = 1.0$ years			
Constant proportion ($p = .5$)	680	626	278	296	.28	.45
Optimal deterministic	679	625	270	307	.27	.45
Time consistent MV (Case 1)	691	680	199	315	.17	.34
Quadratic shortfall	695	757	146	254	.12	.18
			Expected Blocksize $\hat{b} = 2.0$ years			
Constant proportion ($p = .5$)	677	628	264	304	.27	.46
Optimal deterministic	676	625	257	312	.26	.45
Time consistent MV (Case 1)	695	681	200	330	.17	.33
Quadratic shortfall	699	757	137	279	.10	.17
			Expected Blocksize $\hat{b} = 5.0$ years			
Constant proportion ($p = .5$)	675	636	250	313	.27	.44
Optimal deterministic	674	635	246	318	.26	.44
Time consistent MV (Case 1)	701	701	197	348	.15	.33
Quadratic shortfall	708	766	130	310	.09	.16

TABLE 6.7: *Stationary moving block bootstrap resampling results for base case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Strategies are rebalanced annually and are based on the synthetic market with* $E[W_T] = 705.6$ *in all cases. Calculations based on 10,000 bootstrap resamples of historical data for the period 1926:1 to 2015:12.*

					Probability of Shortfall	
Strategy	$E[W_T]$	$Median[W_T]$	$std[W_T]$	CVAR (5%)	$W_T < 700$	$W_T < 900$
Constant proportion ($p = 0.5$)	1085.2	874	860	332	.33	.52
Optimal deterministic	1085.2	878	846	345	.32	.52
Time consistent (Case 1)	1085.2	1029	483	310	.20	.38
Time consistent (Case 2)	1085.2	781	1261	171	.44	.58
Quadratic shortfall	1085.2	1243	342	226	.17	.23

TABLE 6.8: *Synthetic market results from 160,000 Monte Carlo simulation runs for alternative case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2.*

and 5.2 for relevant corresponding parameter estimates. We retain the same assumptions regarding investment horizon, rebalancing frequency, and real cash contributions as for the base case. Using the 10-year Treasury bond index provides a stress test for our assumption of bond process (2.5) with an average long-term rate. As we shall see, when tested on bootstrapped historical data with stochastic bond index returns, our strategy determined using the average long-term bond index return produces statistical results that are very similar to the synthetic market results. This indicates that our parsimonious model formulation is sufficient for generating an investment strategy which is superior to a deterministic strategy.

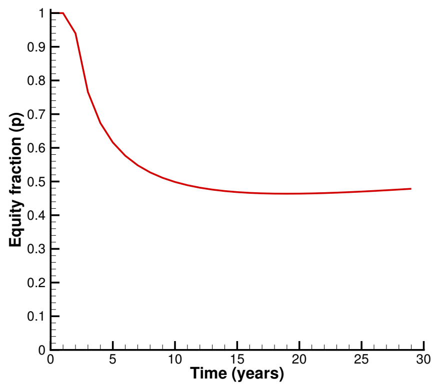
6.4.1 Synthetic Market - Alternative Case

Table 6.8 presents the results for the constant proportion, optimal deterministic, and adaptive strategies. The results are very similar in qualitative terms to those seen earlier for the base case in Table 6.4, though investing in these two assets leads to a terminal wealth distribution with a higher mean, median, and standard deviation relative to using the value-weighted index and 3-month T-bills. We continue to observe that the optimal deterministic strategy barely outperforms a simpler constant weight alternative. The quadratic shortfall strategy continues to be superior to the other strategies by all measures except for the 5% CVAR. Again, note the poor outcomes for the time consistent MV (Case 2) strategy.

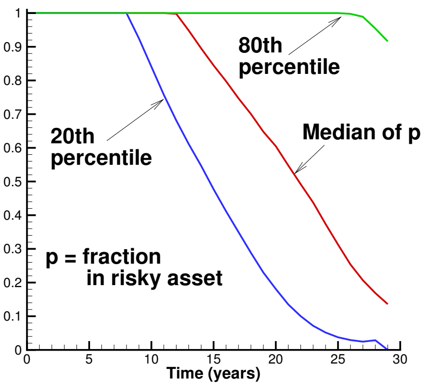
Figures 6.5 and 6.6 show the optimal controls for the various strategies. For the deterministic control in Figure 6.5(a), we focus only on the case with $T = 30$ years. In the case of the adaptive strategies, we show the median as well as the 20th and 80th percentiles of the optimal adaptive control $p(W_t, t)$. In Figure 6.5(b), the quadratic shortfall strategy often departs from the median allocation after about the first decade, reflecting the accumulated wealth from realized returns. As with the value-weighted case, Figure 6.6(a) shows that the time consistent MV (Case 1) controls show a tighter spread about the median, compared with the quadratic shortfall strategy. The time consistent MV (Case 2) controls in Figure 6.6(b) are similar to the the base case, i.e. invest completely in bonds for the first 15 years, followed by a rapid increase to an all equity portfolio by year 20.

6.4.2 Resampled Historical Data - Alternative Case

We use similar bootstrap resampling procedures as described above in Section 6.3.2, but this time for the alternative case with the equal-weight equity and 10-year Treasury indexes. Table 6.9 shows the results for expected blocksizes ranging from 0.25 to 5.0 years. In all cases, the quadratic shortfall

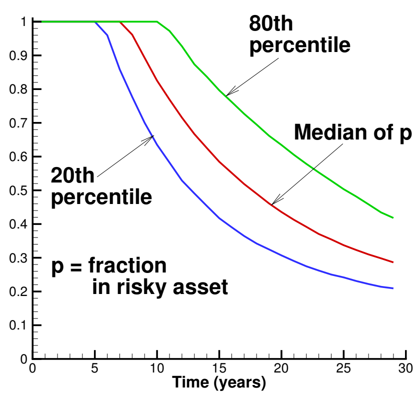


(a) *Optimal deterministic control.*

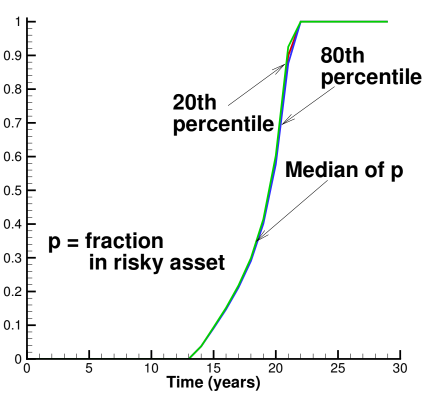


(b) *Quadratic shortfall.*

FIGURE 6.5: *Properties of the strategies using alternative case input data from Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. In each case, $E[W_T]$ is constrained to match that of a constant proportion strategy with $p = 0.5$. Figure 6.5(b) is based on 160,000 Monte Carlo simulation runs.*



(a) *Time consistent MV: Case 1.*



(b) *Time consistent MV: Case 2.*

FIGURE 6.6: *Time consistent MV controls using alternative case input data from Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Synthetic market. $E[W_T]$ is constrained to match that of a constant proportion strategy with $p = 0.5$. 160,000 Monte Carlo simulation runs.*

Strategy	$E[W_T]$	$Median[W_T]$	$std[W_T]$	CVAR (5%)	Probability of Shortfall $W_T < 700$	Probability of Shortfall $W_T < 900$
Expected Blocksize $\hat{b} = 0.25$ years						
Constant proportion ($p = .5$)	1015	863	615	335	.34	.53
Optimal deterministic	1014	865	602	343	.33	.53
Time consistent MV (Case 1)	1040	988	429	329	.21	.41
Quadratic shortfall	1044	1171	316	255	.16	.25
Expected Blocksize $\hat{b} = 0.5$ years						
Constant proportion ($p = .5$)	1005	868	585	337	.33	.53
Optimal deterministic	1004	869	582	346	.33	.53
Time consistent MV (Case 1)	1033	980	421	336	.21	.42
Quadratic shortfall	1041	1163	314	260	.16	.25
Expected Blocksize $\hat{b} = 1.0$ years						
Constant proportion ($p = .5$)	984	864	526	348	.33	.54
Optimal deterministic	982	863	516	356	.32	.54
Time consistent MV (Case 1)	1030	980	402	360	.20	.41
Quadratic shortfall	1046	1155	305	277	.16	.25
Expected Blocksize $\hat{b} = 2.0$ years						
Constant proportion ($p = .5$)	961	865	465	373	.31	.54
Optimal deterministic	959	861	457	379	.31	.54
Time consistent MV (Case 1)	1025	974	377	410	.18	.41
Quadratic shortfall	1064	1148	277	326	.13	.22
Expected Blocksize $\hat{b} = 5.0$ years						
Constant proportion ($p = .5$)	936	869	382	394	.29	.54
Optimal deterministic	936	866	380	397	.29	.54
Time consistent MV (Case 1)	1024	978	340	469	.15	.40
Quadratic shortfall	1090	1155	241	414	.09	.19

TABLE 6.9: *Stationary moving block bootstrap resampling results for alternative case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Strategies are rebalanced annually and are based on the synthetic market with $E[W_T] = 1085.2$ in all cases. Calculations based on 10,000 bootstrap resamples of historical data for the period 1926:1 to 2015:12.*

strategy has a higher median, and lower probabilities of shortfall for $W_T = 700$ and $W_T = 900$. However, the 5% CVAR for the quadratic shortfall strategy is worse than for the other strategies.

Figure 6.7 shows the cumulative distribution functions for the various strategies computed using bootstrap resampling of the historical data. The quadratic shortfall strategy dominates the other strategies for final wealth values greater than about $W_T = 600$ (i.e. about 90% of the time). However, we can see that the other strategies perform better in the extreme left tail. The time consistent MV (Case 1) strategy dominates the deterministic strategies except for very low probability events ($< 1\%$).

7 Misspecified Parameters

As a final robustness check, in Appendix C, we carry out Monte Carlo simulations using parameters different from the strategy generating parameters. This corresponds to computing the strategy based

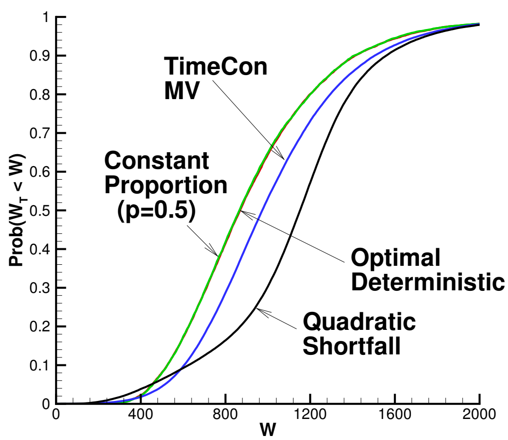


FIGURE 6.7: *Cumulative distribution functions using alternative case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2.* "TimeCon MV" *refers to time consistent MV, Case 1, as in equation (4.6). Distributions are computed using 10,000 bootstrap resamples historical data from 1926:1 to 2015:12. Expected blocksize* $\hat{b} = 2$ *years. Strategies based on the synthetic market with* $E[W_T] = 1085.2$ *in all cases.*

on an incorrect estimate of the stochastic model parameters. In particular, we focus on misspecified values of the stock index drift rate, which is probably the most questionable parameter. The tests show that all methods perform worse under an actual drift rate lower than the assumed drift rate, but the relative rankings of the strategies are preserved.

8 A Portfolio of Employees

We have focused on examining the allocation strategy for a single investor. However, it is straightforward to imagine using these strategies for a company managed DC plan having many heterogeneous employees. Each employee has a different salary (i.e. savings rate), retirement date, median target wealth, and risk attitude (perhaps determined by CVAR preferences). At cash flow dates (e.g. monthly) we can determine the optimal bond and stock amounts for each member of the DC plan, based on her parameters (e.g. current accumulated wealth, retirement date). We then total up all the stock and bond amounts for all members of the DC plan and pool all these amounts together to form a large fund. Each member of the DC plan is then allocated the correct number of units of the pooled stock and bond indexes. This total fund is then rebalanced if necessary. This allows for an efficient, low cost method which allows for a personalized asset allocation strategy for each member of the DC plan.

9 Conclusion

We compare optimal deterministic (open loop) strategies to simpler constant proportion alternatives, based on minimizing the variance of terminal wealth for fixed expected terminal wealth. We find that the best possible deterministic strategy (under MV criteria) gives at most very slight improvement over the simpler constant proportion strategy. Moreover, the efficiency of these strategies is not compromised in any significant way by relatively infrequent (i.e. annual) rebalancing, as opposed to being continuously rebalanced.

We also compare optimal deterministic strategies to adaptive (closed loop) strategies, based on two common suggestions in the literature: time consistent MV and quadratic shortfall. Under both synthetic markets and bootstrap resampling of historical data, we observe that:

- Consistent with the theory, optimal deterministic strategies offer virtually no improvement compared to constant proportion strategies.

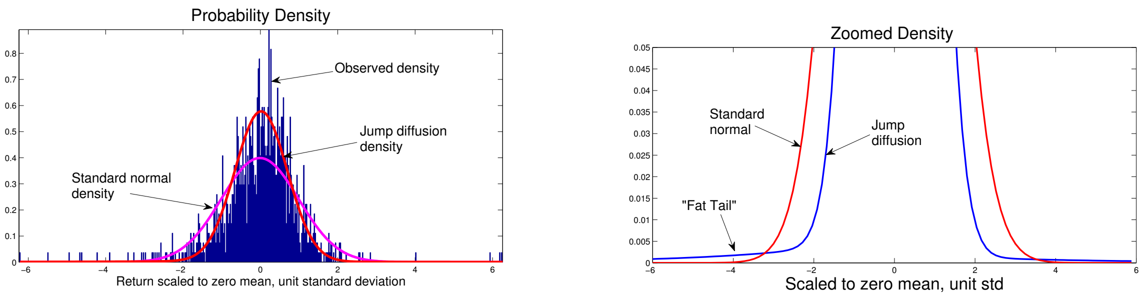
- By most measures, the adaptive strategies outperform the deterministic strategies.

- The time consistent MV objective with wealth-dependent risk aversion performs poorly compared to the time-consistent MV objective with a constant parameter.

- The time consistent MV (constant risk aversion parameter) strategy is superior to the deterministic strategies in terms of median, standard deviation and probability of shortfall. The time consistent MV strategy has a slight decrease in CVAR (5%) (i.e. more risk) compared to the deterministic strategies in some cases.[5]

- Over a wide range of outcomes the quadratic shortfall strategy is superior to the time consistent MV strategy (constant risk aversion parameter), but at the expense of increased left tail risk.

[5]Recall that our definition of CVAR is in terms of final wealth, not losses, so a larger CVAR has less risk.



(a) *Log returns and densities.* (b) *Zoom of Figure A.1(a).*

FIGURE A.1: *Actual and fitted log returns for real CRSP value-weighted index. Monthly data, 1926:1-2015:12, scaled to unit standard deviation and zero mean. Standard normal density and fitted double exponential density (threshold, $\alpha = 3$) also shown.*

- The ranking of the strategies is robust to misspecification of the drift of the stock index.

In short, over the past decade U.S. individuals have invested heavily in TDFs, which are now commonly offered as a default choice. This is a clear improvement over the situation around the turn of the century, where the default allocation was to a money market account. However, our results strongly suggest that TDFs themselves may be far from an optimal solution for investors saving for retirement. Time consistent MV strategies (with a constant risk aversion parameter) are clearly an improvement compared to deterministic strategies. Depending on the investors aversion to extreme tail events, quadratic shortfall is also a viable strategy which is superior to time consistent MV over a wide range of outcomes. Of course, our main conclusion that adaptive strategies are superior to deterministic strategies is based on statistical properties. It is not possible to conclude that an adaptive strategy will beat a deterministic strategy in all future states of the world. However, on average an adaptive strategy is a good bet.

Appendix

A Calibration of Model Parameters

In this Appendix, we discuss the estimation of the parameters of the jump diffusion process given by equations (2.1) and (2.3). Consider a discrete series of index prices $S(t_i) = S_i, i = 1, \ldots, N+1$ that are observed at equally spaced time intervals $\Delta t = t_{i+1} - t_i, \forall i$, with $T = N\Delta t$. We assume equal spacing for simplicity. Given log returns $\Delta X_i = \log\left(S_{i+1}/S_i\right)$, define detrended log returns as $\Delta \hat{X}_i = \Delta X_i - \hat{m}\,\Delta t$, where $\hat{m} = \left[\log\left(S_{N+1}\right) - \log\left(S_1\right)\right]/T$.

Figure A.1(a) shows a histogram of the monthly log returns from the real value-weighted CRSP total return index, scaled to zero mean and unit standard deviation. We superimpose a standard normal density onto this histogram. We also superimpose the fitted density for the double exponential jump diffusion model. The plot shows that the empirical data is leptokurtic, having a higher peak and fatter tails than a normal distribution, consistent with previous empirical findings for virtually all financial time series. Figure A.1(b) zooms in on these two densities, to better reveal the fat tails of the jump diffusion model.

A standard technique for parameter estimation is maximum likelihood (ML). However, it is well-known that the use of ML estimation for a jump diffusion model is problematic, due to multiple local maxima and the ill-posedness of trying to distinguish high frequency small jumps from diffusion

(Honore, 1998). Alternative econometric techniques have been developed for detecting the presence of jumps in high frequency data, i.e. on a time scale of seconds (Aït-Sahalia and Jacod, 2012). However, from the perspective of a long-term investor, the most important feature of a jump diffusion model is that it allows modelling of infrequent large jumps in asset prices. Small and frequent jumps look like enhanced volatility when examined on a large scale, hence these effects are probably insignificant when constructing a long-term investment strategy. Consequently, as an alternative to ML estimation, we use the thresholding technique described in Mancini (2009) and Cont and Mancini (2011). This procedure is considered to be more efficient for low frequency data.

Suppose we have an estimate for the diffusive volatility component $\hat{\sigma}$. Then we detect a jump in period i if

$$\left|\Delta \hat{X}_i\right| > \mathcal{A}\, \hat{\sigma}\, \frac{\sqrt{\Delta t}}{(\Delta t)^{\beta}} \tag{A.1}$$

where $\beta, \mathcal{A} > 0$ are tuning parameters (Shimizu, 2013), and $\hat{\sigma}$ is our most recent estimate of volatility. An iterative method is used to determine the parameters (Clewlow and Strickland, 2000). The intuition behind equation (A.1) is simple. If we choose $\mathcal{A} = 3$, say, and $\beta \ll 1$, then equation (A.1) identifies an observation as a jump if the observed log return exceeds a 3 standard deviation geometric Brownian motion change. Typically, β in equation (A.1) is quite small, $\beta \simeq .01 - .02$. For details, we refer the reader to Dang and Forsyth (2016). As described in Dang and Forsyth (2016), we replace $\mathcal{A}/(\Delta t)^{\beta}$ by the parameter α. Use of $\alpha = 3$ for monthly data results in fairly infrequent, large jumps. Additional details concerning the ML and threshold estimators can be found in Dang and Forsyth (2016) and Forsyth and Vetzal (2017).

B Bootstrap Resampling

Suppose we are given a sequence of historical returns ΔX_i, $i = 1, \ldots, N$. We wish to draw a sequence of $N_s < N$ returns from this historical sequence, using a stationary block bootstrap resampling technique (Politis and Romano, 1994).

Given an estimate of the expected blocksize (Patton et al., 2009), we use Algorithm B.1 to produce a sequence of returns of length N_s from the historical return sequence of length N. Note that we use a circular block bootstrap, i.e. the historical sequence is wrapped around if necessary. Algorithm B.1 produces a single sequence of returns of the required length. This algorithm is then called N_{sim} times to generate statistics based on N_{sim} bootstrap resamples.

For ease of exposition, we have written Algorithm B.1 as taking as input a single sequence of historical returns and producing a single resampled sequence of output returns. In our application, we simultaneously sample from the two historical stock and bond sequences of returns, and output a paired set of stock and bond resampled returns.

C Robustness to Misspecified Parameters

In this Appendix, we test the robustness of the better performing strategies to misspecified parameters. We compute and store the optimal controls as usual in the synthetic market, using the parameters in Tables 5.1 (threshold) and 5.2. We then carry out Monte Carlo simulations the using stored controls, except that in these simulations we reduce the drift rate μ (equation (2.3)) of the stock process by 200 basis points. This corresponds to using an incorrect (optimistic) estimate of the drift rate for stocks.

Require: Function geo($\hat{b}$) ; returns draw from a shifted geometric distribution with mean $\hat{b}$
Require: Function rand_int(N) ; returns a uniformly distributed draw from $\{1,2,\ldots,N\}$

input: Vector of returns ΔX_i ; $i = 1,\ldots,N$; Expected block size $\hat{b}$; Number of samples N_s
initialize: actual_block_size = 0 ; total_samples = 1 ; sub_block_total = N

while (total_samples $\leq N_s$) **do**
  if (sub_block_total > actual_block_size) **then**
    actual_block_size = geo($\hat{b}$) {restart subblock}
    index = rand_int(N) {choose random starting index}
    sub_block_total = 1
  end if
  index = mod(index, N) {circular bootstrap}
  out_array(total_samples) = ΔX_{index}
  index += 1 ; total_samples += 1 ; sub_block_total += 1
end while

return out_array(i) ; $i = 1,\ldots,N_s$

Algorithm B.1: *A single stationary block bootstrapped sequence of returns.*

Strategy	$E[W_T]$	$Median[W_T]$	$std[W_T]$	CVAR (5%)	Probability of Shortfall $W_T < 500$	$W_T < 600$
Constant proportion ($p = 0.5$)	580	522	272	250	.46	.64
Time consistent MV (Case 1)	605	588	214	228	.32	.52
Quadratic shortfall	645	718	197	167	.24	.33

Table C.1: *Synthetic market results from 160,000 Monte Carlo simulation runs. Strategy computed using base case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Monte Carlo simulations carried out reducing the stock asset drift μ (equation (2.3)) by 200bps. Effective real stock index geometric return* .045.

The results for the base case are shown in Table C.1 and for the alternate case in Table C.2 Obviously, all the statistics for all strategies are worse than for the unreduced drift simulations. However, the relative rankings of all the strategies are preserved. The adaptive strategies perform better than the the constant weight strategy, with the exception of the CVAR (5%) statistic. In addition, the quadratic shortfall statistics are superior to the time consistent MV (case 1) strategy with the exception of CVAR.

Recall that tests in Ma and Forsyth (2016) show that the effect of a mean reverting stochastic volatility process is negligible for a long-term investor, with typical historical mean reversion speeds. The tests were carried out using the pre-commitment mean variance objective function. It is interesting to see the effect of a more extreme situation, where we allow the volatility to be drawn from a uniform distribution in the range $[\sigma_{min}, \sigma_{max}]$. Table C.3 shows the Monte Carlo simulation results for the alternate case data. The simulation results are virtually indistinguishable from the constant volatility case.

Strategy	$E[W_T]$	$Median[W_T]$	$std[W_T]$	CVAR (5%)	Probability of Shortfall $W_T < 700$	Probability of Shortfall $W_T < 900$
Constant proportion ($p = 0.5$)	876	716	647	283	.48	.67
Time consistent (Case 1)	927	870	458	212	.33	.53
Quadratic shortfall	953	1099	396	160	.29	.37

TABLE C.2: *Synthetic market results from 160,000 Monte Carlo simulation runs. Strategy computed using the alternative case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Monte Carlo simulations carried with stock asset drift μ (equation (2.3)) reduced by 200 bps. Effective real stock index geometric return* .06.

Strategy	$E[W_T]$	$Median[W_T]$	$std[W_T]$	CVAR (5%)	Probability of Shortfall $W_T < 700$	Probability of Shortfall $W_T < 900$
Constant proportion ($p = 0.5$)	1079	873	824	331	.34	.52
Time consistent (Case 1)	1082	1027	483	309	.20	.38
Quadratic shortfall	1082	1243	342	223	.17	.24

TABLE C.3: *Synthetic market results from 160,000 Monte Carlo simulation runs. Strategy computed using the alternative case input data given in Table 6.1 and corresponding parameters from Tables 5.1 (threshold) and 5.2. Monte Carlo simulations carried with volatility σ drawn from a uniform distribution $\sigma \in [.09256, .19256]$, at each rebalancing date.*

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Declarations of Interest

The authors have no conflicts of interest to declare.

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