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On the Distribution of Terminal Wealth under Dynamic Mean-Variance Optimal Investment Strategies*

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Abstract. We compare the distributions of terminal wealth obtained from implementing the optimal investment strategies associated with the different approaches to dynamic mean-variance (MV) optimization available in the literature. This includes the precommitment MV (PCMV) approach, the dynamically optimal MV (DOMV) approach, as well as the time-consistent MV approach with a constant risk aversion parameter (cTCMV) and wealth-dependent risk-aversion parameter (dTCMV), respectively. For benchmarking purposes, a constant proportion (CP) investment strategy is also considered. To ensure that terminal wealth distributions are compared on a fair and practical basis, we assume that an investor, otherwise agnostic about the philosophical differences of the underlying approaches to dynamic MV optimization, requires that the same expected value of terminal wealth should be obtained regardless of the approach. We present first-order stochastic dominance results proving that for wealth outcomes below the chosen expected value target, the cTCMV strategy always outperforms the DOMV strategy, and an appropriately chosen CP strategy always outperforms the dTCMV strategy. We also show that the PCMV strategy results in a terminal wealth distribution with fundamentally different characteristics than any of the other strategies. Finally, our analytical results are very effective in explaining the numerical results currently available in the literature regarding the relative performance of the various investment strategies.

Key words. asset allocation, constrained optimal control, time-consistent, mean-variance

AMS subject classifications. 91G, 65N06, 65N12, 35Q93

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1. Introduction.

Originating with Markowitz (1952), mean-variance (MV) portfolio optimization forms the foundation of modern portfolio theory (Elton et al. (2014)), in part due to its intuitive nature. In dynamic settings (see, for example, Zhou and Li (2000)), MV optimization aims to obtain an investment strategy that maximizes the expected value of the terminal wealth of the portfolio, for a given level of risk as measured by the associated variance of the terminal wealth.

It is well known that variance does not satisfy the law of iterated expectations. As a result, the MV objective is not separable in the sense of dynamic programming, resulting in three main approaches to MV optimization that can be identified in the literature.

In the first approach, referred to as precommitment MV (PCMV) optimization, the

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resulting optimal investment strategy is typically time-inconsistent when viewed from the perspective of the original MV objective (Basak and Chabakauri (2010)). However, in practice the PCMV problem is solved using the embedding approach of Li and Ng (2000) and Zhou and Li (2000), and the resulting PCMV-optimal investment strategy is time-consistent from the perspective of the induced quadratic objective function used in the corresponding embedding problem (Vigna (2014, 2020)). Therefore, the PCMV-optimal investment strategies considered in this paper are in fact feasible to implement as trading strategies (see Strub, Li, and Cui (2019)).

The second approach, referred to as time-consistent MV (TCMV) optimization, is based on a game-theoretic approach to the MV problem (Basak and Chabakauri (2010); Bjork and Murgoci (2014)). The TCMV-optimal investment strategies are guaranteed to be time-consistent, since optimization is performed only over a subset of investment strategies which are time-consistent from the perspective of the original MV problem. Equivalently, in the TCMV approach the MV problem is solved subject to a time-consistency constraint on the admissible investment strategies (Cong and Oosterlee (2016a); Wang and Forsyth (2011)). Two main variations of the TCMV approach can be found in the literature, depending on the treatment of the risk-aversion parameter which encodes the investor's risk preferences in an MV setting. Specifically, the risk-aversion parameter is either assumed to be a constant over the entire investment time horizon (see, for example, Basak and Chabakauri (2010)), or it is assumed to be "wealth-dependent," in particular, inversely proportional to the investor's wealth at any given point in time (Bjork, Murgoci, and Zhou (2014)). To distinguish between these two cases, we refer to the TCMV approach that uses a constant risk-aversion parameter as the cTCMV approach, and to the case that uses the wealth-dependent risk-aversion parameter as the dTCMV approach.

The third approach, namely the dynamically optimal MV (DOMV) optimization approach of Pedersen and Peskir (2017), entails solving an infinite number of problems with the MV objective dynamically forward in time. In particular, starting from an initial wealth and initial time, each new wealth level attained over time results in a new MV problem that has to be solved, resulting in a new optimal strategy to be implemented only at that time instant and for that particular wealth level. The resulting DOMV-optimal strategy therefore differs fundamentally from the TCMV-optimal strategy but is indeed feasible to implement as a trading strategy.

We briefly note that each of these approaches to dynamic MV optimization is associated with a different underlying motivational philosophy. In this sense, preference for one strategy over another depends on the MV investor's investment philosophy and perspective on time-consistency; see Vigna (2017, 2020) for a number of the subtle issues involved. However, for a practical assessment of the relative performance of the different investments strategies, we do not dwell on these philosophical considerations in this paper and instead only focus on wealth outcomes.

Recently, dynamic MV optimization has received considerable attention in institutional settings, including pension fund and insurance applications; see, for example, Chen, Li, and Guo (2013); Forsyth and Vetzal (2019a); Forsyth, Vetzal, and Westmacott (2019); Højgaard and Vigna (2007); Liang, Bai, and Guo (2014); Lin and Qian (2016); Menoncin and Vigna (2013); Nkeki (2014); Sun, Li, and Zeng (2016); Vigna (2014); Wang and Chen (2018, 2019);

Wei and Wang (2017); Wu and Zeng (2015); Zhao, Shen, and Zeng (2016); and Zhou et al. (2016), among many others. In particular, we also highlight the popularity of the dTCMV approach in institutional settings, for example, in the case of the investment-reinsurance problems faced by insurance providers (Bi and Cai (2019); Li and Li (2013)), investment strategies for pension funds (Liang, Bai, and Guo (2014); Sun, Li, and Zeng (2016); Wang and Chen (2018, 2019)), corporate international investment (Long and Zeng (2016)), and asset-liability management (Peng, Cui, and Shi (2018); Zhang et al. (2017)).

In all of these situations, it is reasonable to argue that the *distribution* of terminal wealth is of key importance to stakeholders, despite the natural focus in the literature on the mean and variance of terminal wealth. The reason for this is that in any practical setting, the MV investor (or indeed, any investor) is likely to also take into account a number of other measures of risk and investment performance,[1] which might be critical even if only as a result of regulatory considerations (see, for example, Antolin et al. (2009)). As noted in Goetzmann et al. (2002), in a complete market, a dynamic trading strategy can be viewed as a strategy consisting of the risky asset and options written on this asset. This changes the final wealth distribution from a standard log-normal distribution (in the Black–Scholes market) in a nontrivial manner. Hence, even if we consider "Sharpe ratio" maximizing strategies, it is of interest to examine other properties (e.g., skewness, kurtosis) of the terminal wealth distribution.

In light of these considerations, it is therefore not surprising that there has been significant interest recently in different aspects of the terminal wealth distribution obtained under various investment strategies, including optimal strategies associated with approaches to dynamic MV portfolio optimization; see, for example, Forsyth and Vetzal (2017a,b, 2019a,b); Forsyth, Vetzal, and Westmacott (2019). These papers present a very realistic formulation of the underlying problems, including, for example, the treatment of withdrawals and contributions, investment constraints, and so on. By necessity, these papers therefore focus on the results obtained from the numerical solutions of the problems under consideration.

In contrast, there seems to be very little available research on the theoretical comparison of the terminal wealth distributions in cases where the optimal investment strategies can be expressed analytically. We emphasize that while analytical MV-optimal strategies sometimes call for unacceptably high leverage ratios or unrealistic treatment of insolvency, investment constraints can be incorporated easily in the numerical solution of the MV optimization problem (see, for example, Cong and Oosterlee (2016b); Dang and Forsyth (2014); Van Staden, Dang, and Forsyth (2018); Wang and Forsyth (2010, 2011)). However, analytical investment strategies remain very useful, in that an analytical comparison of terminal wealth distributions (i) can provide an additional perspective on some of the implications of the various approaches to dynamic MV optimization that is currently not available in the literature, and (ii) can assist in explaining some of the numerical results recently reported in the literature, such as

[1]We observe that it is possible for an investor to explicitly incorporate additional risk and/or performance criteria as part of the objective function, instead of simply performing MV optimization. For example, portfolio optimization with higher-order moments can be performed (see, for example, Jurczenko, Maillet, and Merlin (2012) and Maringer and Parpas (2009)). However, as the MV objective remains by far the most popular objective function in the recent dynamic portfolio optimization literature, we correspondingly focus on the case of MV optimization, leaving other formulations for our future work.

the results of, for example, Forsyth and Vetzal (2017b) and Forsyth, Vetzal, and Westmacott (2019).

The main objective of this paper is therefore a systematic comparison of the analytical terminal wealth distributions resulting from the optimal investment strategies associated with the different approaches to dynamic MV optimization in the literature. In order to compare distributions on a fair basis, we assume that the investor remains agnostic as to the philosophical differences underlying the various approaches to MV optimization, and simply wishes to achieve a chosen expected value of terminal wealth regardless of the approach. Our main contributions are as follows:

- We derive analytical results regarding the terminal wealth distributions that, despite our assumption of no market frictions (in particular, continuous trading with no leverage constraints, no transaction costs, and without insolvency/bankruptcy prohibitions), are very effective in explaining the numerical results incorporating realistic investment constraints currently available in the literature.
- For comparison and benchmarking purposes, our analysis includes a simple constant proportion (CP) strategy, whereby the investor invests a fixed proportion of wealth in the risky asset throughout the investment time horizon. The CP strategy is typically not MV-optimal in the sense of any of the other strategies considered, but our analysis proves that it easily outperforms the dTCMV-optimal investment strategy in the general sense of a partial first-order stochastic dominance result we present.
- Our results also show that the dTCMV-optimal strategy performs exceptionally poorly compared to the other MV-optimal investment strategies, with, for example, the dTCMV-optimal strategy achieving both a higher variance and lower median terminal wealth than the cTCMV strategy. This calls into question the current popularity enjoyed by the dTCMV-optimal strategy in the literature.
- We establish that the cTCMV strategy outperforms the DOMV strategy in a first-order stochastic dominance sense when we consider terminal wealth outcomes below the expected value target. The cTCMV strategy also achieves a lower variance of terminal wealth compared to the DOMV strategy.
- Furthermore, we derive analytical results which prove that the PCMV strategy results in a terminal wealth distribution with fundamentally different characteristics than any of the other strategies. In particular, the PCMV-optimal strategy achieves the lowest variance and highest median value of terminal wealth of all the strategies considered, but the negative skewness and large kurtosis of the associated terminal wealth distribution means that the otherwise excellent performance of the PCMV strategy comes at the cost of increased left tail risk for the investor.
- Numerical results, making use of model parameters calibrated to inflation-adjusted, long-term US market data (1926–2014), are presented to validate and illustrate the implications of our analytical results.

The remainder of the paper is organized as follows. Section 2 describes the underlying dynamics, notational conventions, and rigorous definitions of the different approaches to dynamic MV optimization. Subject to certain assumptions, section 3 presents a number of analytical results, including some new results, regarding the terminal wealth distributions associated with different approaches. In section 4, we present a rigorous analytical comparison study of terminal wealth distributions associated with different approaches, but all achieve the investor's chosen expected value target. Numerical results are presented in section 5, while section 6 concludes the paper and outlines possible future work.

2. Formulation. For simplicity, our analysis focuses on portfolios consisting of a well-diversified stock index (the risky asset) and a risk-free asset. Since the available analytical solutions for multiasset PCMV and TCMV approaches (see, for example, Li and Ng (2000) and Zeng and Li (2011)) show that the overall composition of the risky asset basket remains relatively stable over time, it is reasonable to focus on the overall risky asset basket vs. risk-free asset composition of the portfolio as the primary investment question. We leave the extension of our results to multiasset dynamic MV optimization problems for our future work.

Let $t_0 \equiv 0$ denote the start of the investment time period, and let $T > 0$ denote the fixed investment time horizon or maturity. The controlled wealth, with the control representing some investment strategy, is denoted by $W(t)$, $t \in [t_0, T]$. Specifically, let $u : (W(t), t) \mapsto u(t) = u(W(t), t)$, $t \in [t_0, T]$, be the adapted feedback control representing the amount invested in the risky asset at time t given wealth $W(t)$, and let $\mathcal{A} = \{u(t) = u(w,t) | u : \mathbb{R} \times [t_0, T] \to \mathbb{U}\}$ denote the set of admissible controls, where $\mathbb{U} \subseteq \mathbb{R}$ denotes the admissible control space.

We assume that the risky asset follows a geometric Brownian motion (GBM), leaving the treatment of jumps in the risky asset process and alternative model specifications for our future work. While this choice of model may appear to be overly simplistic, we observe the following: (i) The extensive backtesting results presented in Forsyth and Vetzal (2017b) show that the GBM assumption actually performs very well over long investment time horizons, suggesting that more complicated models (including, for example, incorporating stochastic volatility (Ma and Forsyth (2016)) may not offer substantial advantages in this setting. (ii) As discussed in more detail below, the analytical results presented in this paper (based on GBM dynamics) are in qualitative agreement with the numerical results presented in Forsyth and Vetzal (2019b) and Forsyth, Vetzal, and Westmacott (2019) where jump-diffusion models are assumed for the risky asset, indicating that a GBM model appears to be sufficient in capturing the salient characteristics of the different investment strategies.

Therefore, based on the assumption of GBM dynamics for the risky asset, the dynamics of the wealth $W(t)$ of a self-financing portfolio, with no contributions or withdrawals, is given by (see, for example, Bjork (2009); Bjork, Murgoci, and Zhou (2014))

$$dW(t) = [rW(t) + (\mu - r)\, u(t)]\, dt + \sigma u(t)\, dZ(t), \quad t \in (t_0, T], \tag{2.1}$$

$$W(t_0) = w_0 > 0. \tag{2.2}$$

Here, $w_0 > 0$ denotes the initial wealth, $r > 0$ denotes the continuously compounded risk-free interest rate, $\mu > r$ and $\sigma > 0$ denote the drift and volatility of the dynamics of the risky asset, respectively, and Z denotes a standard Brownian motion. For subsequent reference, we also define the following combination of parameters:

$$A = \frac{(\mu - r)^2}{\sigma^2}. \tag{2.3}$$

Before presenting rigorous definitions of the various approaches to dynamic MV optimization, we introduce a number of notational conventions. Let $Q_u^{w,t}[W(T)]$ denote some quantity Q

associated with the terminal wealth $W(T)$, given wealth $W(t) = w$ at time $t \in [0,T]$ and the application of control $u \in \mathcal{A}$ over the time interval $[t,T]$. Specific examples of the quantity Q encountered in this paper include the expected value (in which case we set $Q = E$), variance ($Q = Var$), standard deviation ($Q = Stdev$), conditional probability measure ($Q = \mathbb{P}$), and the value-at-risk and conditional value-at-risk[2] at level $\alpha \in (0,1)$, respectively denoted by $Q = \alpha$VaR and $Q = \alpha$CVaR. The optimal control and optimal terminal wealth will be denoted by u_j^* and $W_j(T)$, respectively, where the subscript $j \in \{p, d, c, cd, cp\}$ is used to distinguish the underlying approach with respect to which u_j^* and $W_j(T)$ are optimal. For ease of subsequent reference, the particular association of the subscript j with the corresponding investment approach is outlined in Table 2.1.

Table 2.1

Summary of notational conventions. The subscript $j \in \{p, d, c, cd, cp\}$ is used to identify the approach in terms of which the optimal investment strategy u_j^* and associated optimal terminal wealth $W_j(T)$ is obtained. For the sake of simplicity, the CP strategy is identified using similar notation, but we emphasize that the CP strategy is not formulated to be MV-optimal.

Subscript j	Approach	Abbreviation	Optimal control u_j^*	Optimal terminal wealth using control u_j^*
$j = p$	Precommitment MV	PCMV	u_p^*	$W_p(T)$
$j = d$	Dynamically optimal MV	DOMV	u_d^*	$W_d(T)$
$j = c$	Time-consistent MV with constant risk-aversion parameter	cTCMV	u_c^*	$W_c(T)$
$j = cd$	Time-consistent MV with wealth-dependent risk-aversion parameter	dTCMV	u_{cd}^*	$W_{cd}(T)$
$j = cp$	Constant proportion strategy	CP	u_{cp}^*	$W_{cp}(T)$

We now present the definitions of the main approaches to MV portfolio optimization considered in this paper. Using the standard scalarization method for multicriteria optimization problems (Yu (1974)), a general definition of the dynamic MV optimization problem is given by (see, for example, Zhou and Li (2000))

$$\sup_{u \in \mathcal{A}} \left(E_u^{w_0,t_0}[W(T)] - \rho \cdot Var_u^{w_0,t_0}[W(T)] \right), \qquad \rho > 0, \tag{2.4}$$

where the the investor's level of risk aversion is reflected by the risk-aversion (or scalarization) parameter $\rho > 0$.

As noted in the introduction, variance does not satisfy the smoothing property of conditional expectation, and therefore dynamic programming cannot be applied directly to (2.4). The first approach to dynamic MV optimization, the precommitment MV (PCMV) approach, employs the technique of Li and Ng (2000) and Zhou and Li (2000) to embed problem (2.4) in a new optimization problem, often referred to as the embedding problem, which can be solved using dynamic programming techniques. We follow the convention in the literature (see, for example, Cong and Oosterlee (2017); Dang, Forsyth, and Vetzal (2017)) of defining

[2]The terms and risk measures are defined rigorously below; see section 4.

the PCMV optimization problem as the associated MV embedding problem, namely

$$(PCMV\left(\gamma\right)): \quad \inf_{u\in\mathcal{A}}\left(E_u^{w_0,t_0=0}\left[\left(W\left(T\right)-\frac{\gamma}{2}\right)^2\right]\right), \qquad \frac{\gamma}{2}>w_0e^{rT}, \tag{2.5}$$

where the embedding parameter γ is assumed to satisfy $\gamma > 2w_0e^{rT}$ to ensure that financially meaningful results are obtained (see Dang and Forsyth (2016); Vigna (2014)). As per Table 2.1, we use the notation u_p^* and $W_p\left(T\right)$ to denote the optimal control and optimal terminal wealth for problem (2.5), respectively.

Remark 2.1 (time-consistency of PCMV-optimal control u_p^*.). As discussed in detail in Li and Forsyth (2019) and Forsyth, Vetzal, and Westmacott (2019), there appears to be some confusion in the literature as to whether the PCMV-optimal control u_p^* is time-consistent or not. This question is of great practical significance, since u_p^* is typically time-inconsistent (see Basak and Chabakauri (2010); Bjork and Murgoci (2014)) from the perspective of the original MV objective (2.4), which raises questions regarding its feasibility as an implementable trading strategy. This observation is arguably the reason why a number of different approaches to dynamic MV optimization have been developed, each with a different underlying philosophy as to how the problem of time-inconsistency with respect to the original objective (2.4) is to be addressed; see Vigna (2017, 2020) for a discussion of the various considerations involved. For purposes of clarity, we make a number of observations regarding this issue.

Using the same assumptions as in this paper (including the dynamics (2.1) and the assumptions introduced below in section 3), Vigna (2014) builds on the results of Zhou and Li (2000) to show that there is a *one-to-one* correspondence between the results (including optimal control and the MV efficient frontier) of problems (2.4) and (2.5), provided that ρ in (2.4) at $t_0 = 0$ is related to γ in (2.5) by the relationship

$$\rho = \frac{e^{AT}}{2\left(\frac{\gamma}{2}-w_0e^{rT}\right)}. \tag{2.6}$$

Note that the exact relationship (2.6) between ρ and γ, including its one-to-one nature, might no longer hold if, for example, jumps are included in the wealth dynamics (see Dang, Forsyth, and Li (2016) for a detailed treatment). That said, the key embedding result from Zhou and Li (2000) and Li and Ng (2000) can be shown to hold regardless of the specification of the admissible set of the controls (Dang and Forsyth (2016)).

Therefore, given that the one-to-one relationship (2.6) holds on the basis of the assumptions of this paper, whether we use formulation (2.4) or (2.5) as our starting point does not affect any of the subsequent results, regardless of one's philosophical preference. However, from an investor's perspective, the starting point has important practical consequences. First, Vigna (2014) points out that specifying the "quadratic target" $\gamma/2$ in (2.5) is far more "user-friendly" than specifying ρ in (2.4), since the literature does not offer much guidance as to how ρ should be selected. Second, it is worth emphasizing that, for a fixed value of γ in (2.5), the optimal control u_p^* of (2.5) is a *time-consistent* control from the perspective of the quadratic objective function in (2.5) and is therefore feasible to implement as a trading strategy (see Strub, Li, and Cui (2019)), whereas formulating this control in terms of ρ results in a time-inconsistent (and therefore impractical) trading strategy from the perspective of (2.4).

As a result, it should be clear from this discussion that the issue of the time-consistency of u_p^* is a matter of perspective, and in this paper we always view u_p^* as the time-consistent strategy minimizing the induced objective function in (2.5), and correspondingly formulate all our results in terms of γ. To be precise, the control for the time-inconsistent problem (2.4), for a given value of ρ, specified at time t_0, is identical to the control for time-consistent problem (2.5), with fixed γ given from (2.6). Since this control is the solution of time-consistent problem (2.5), it is a valid or implementable control for all $t \geq t_0$. This treatment aligns with our stated objective of comparing terminal wealth distributions from the perspective of an investor who remains agnostic as to the underlying philosophical differences of the various approaches to dynamic MV optimization.

Next, we consider the dynamically optimal MV (DOMV) approach proposed by Pedersen and Peskir (2017). Informally, this entails solving an infinite number of problems of the form (2.4) dynamically forward in time. Starting from the initial state and time (w_0, t_0), each new state $(W(t), t), t \in [t_0, T]$, attained by the controlled wealth process results in a new problem (2.4) to be solved to obtain the optimal control $u_d^*(W(t), t) := u_d^*(t)$ applicable at that time instant. In this way, the dynamically optimal control $u_d^*(t)$ is obtained for all $t \in [t_0, T]$, resulting in a DOMV-optimal terminal wealth $W_d(T)$. More formally, following Pedersen and Peskir (2017), we define the DOMV problem and associated optimal control u_d^* as follows:

$$
\begin{aligned}
(DOMV(\rho)): \ & u_d^* \in \mathcal{A} \text{ is dynamically optimal for (2.4) with a given fixed } \rho > 0, \\
& \text{if } \forall (w,t) \in \mathbb{R} \times [t_0, T],\ \exists u \in \mathcal{A} \text{ satisfying } u(w,t) = u_d^*(w,t), \\
& \text{such that } \forall v \in \mathcal{A} \text{ with } v(w,t) \neq u_d^*(w,t), \text{we have} \\
& E_u^{w,t}[W(T)] - \rho \cdot Var_u^{w,t}[W(T)] \geq E_v^{w,t}[W(T)] - \rho \cdot Var_v^{w,t}[W(T)]. \qquad (2.7)
\end{aligned}
$$

The time-consistent MV (TCMV) approach (Basak and Chabakauri (2010)) involves maximizing the objective of (2.4) subject to a time-consistency constraint (see, for example, Cong and Oosterlee (2016a); Van Staden, Dang, and Forsyth (2019); Wang and Forsyth (2011)), so that the resulting optimal control is time-consistent from the perspective of the original MV objective (2.4). As noted in the introduction, we distinguish two variants of the TCMV approach, depending on the treatment of the risk-aversion parameter ρ in (2.4).

First, using a constant risk-aversion parameter $\rho > 0$ in (2.4), we define the cTCMV problem as

$$
(cTCMV(\rho)): \quad \sup_{u \in \mathcal{A}} \left(E_u^{w_0,t_0}[W(T)] - \rho \cdot Var_u^{w_0,t_0}[W(T)] \right), \qquad \rho > 0, \tag{2.8}
$$

$$
\text{s.t. } u_c^*(t_0; y, v) = u_c^*(t'; y, v) \quad \text{for } v \geq t', t' \in [t_0, T], \tag{2.9}
$$

where $u_c^*(t_0; y, v)$ denotes the optimal control calculated at time t_0 and to be applied at some future time $v \geq t' \geq t_0$ given future state $W(v) = y$, while $u_c^*(t'; y, v)$ denotes the optimal control calculated at some future time $t' \in [t_0, T]$, also to be applied at the same later time $v \geq t'$ given the same future state $W(v) = y$. To lighten notation, as per Table 2.1 we will use the notation $u_c^*(t)$ to denote the optimal control of the cTCMV problem (2.8)–(2.9).

A popular alternative formulation of the TCMV problem is to specify a risk-aversion parameter that is inversely proportional to wealth; see Bjork, Murgoci, and Zhou (2014) for the motivation and a detailed analysis. Specifically, in this formulation, the constant ρ in (2.8) is replaced by $\rho(w) = \rho/(2w)$ for $\rho > 0$, where w denotes the current wealth. This results in the dTCMV problem defined by

$$(dTCMV(\rho)): \quad \sup_{u\in\mathcal{A}} \left(E_u^{w_0,t_0}[W(T)] - \frac{\rho}{2w_0} \cdot Var_u^{w_0,t_0}[W(T)] \right), \qquad \rho > 0, \tag{2.10}$$

$$\text{s.t. } u_{cd}^*(t_0; y, v) = u_{cd}^*(t'; y, v) \quad \text{for } v \geq t', t' \in [t_0, T], \tag{2.11}$$

where the time-consistency constraint (2.11) has the same interpretation as in (2.9). As per Table 2.1, we denote the dTCMV-optimal control by $u_{cd}^*(t)$ and the associated optimal terminal wealth by $W_{cd}(T)$.

Finally, for benchmarking and comparison purposes, we also consider the constant proportion (CP) problem, defined as follows:

$$\begin{aligned} (CP(\theta_{cp})): &\text{ Choose a constant proportion } \theta_{cp} > 0 \text{ of wealth} \\ &\text{ to invest in the risky asset } \quad \forall t \in [t_0, T], \text{ so that} \\ &\; u_{cp}^*(t) = \theta_{cp} W(t) \quad \forall t \in [t_0, T]. \end{aligned} \tag{2.12}$$

As noted in the introduction, the CP strategy is not designed to be MV-optimal in any sense. However, as per Table 2.1, for convenience we use the notation $u_{cp}^*(t)$ and $W_{cp}(T)$, respectively, to denote the control and terminal wealth associated with the CP problem for some choice of the constant proportion θ_{cp}. A concrete example of choosing a value of θ_{cp} to achieve a specific goal is given in section 4.

3. Selected analytical results.

In this section, we present analytical results relevant to the terminal wealth distributions obtained under the optimal investment strategies of the problems presented in section 2. All results in this section are based on the assumption of no market frictions or investment constraints, which are formally defined as Assumption 3.1.

Assumption 3.1 (no market frictions). *Trading continues in the event of insolvency, no transaction costs are applicable, and no leverage constraints are in effect.*

Remark 3.1 (relaxing Assumption 3.1). Since the simultaneous application of multiple realistic investment constraints can be incorporated with relative ease in the numerical solution of dynamic MV optimization problems (see Cong and Oosterlee (2016b); Dang and Forsyth (2014); Van Staden, Dang, and Forsyth (2018); Wang and Forsyth (2010, 2011), among others), relaxing Assumption 3.1 is not challenging in a practical setting. However, as noted in the introduction, this paper focuses on a theoretical comparison of optimal terminal wealth distributions in the particular select cases where the optimal investment strategies associated with dynamic MV optimization problems can be expressed analytically. The two main consequences of Assumption 3.1 are therefore that it (i) ensures that an additional perspective on the implications of the various approaches to dynamic MV optimization can be presented in this paper that is currently missing from the literature, and (ii) assists in explaining some of the numerical results reported in the literature (see, for example, Forsyth and Vetzal (2017a,b, 2019a,b); Forsyth, Vetzal, and Westmacott (2019)).

Under Assumption 3.1, the optimal controls associated with the dynamic MV optimization problems presented in section 2 can be expressed analytically, as the following lemma shows.

Lemma 3.2 (optimal controls). *Under Assumption 3.1, the optimal controls of problems PCMV (2.5), DOMV (2.7), cTCMV (2.8)–(2.9), and dTCMV (2.10)–(2.11) are, respectively, given by*

$$u_p^*(t) = \frac{A}{(\mu - r)} e^{-r(T-t)} \left[\frac{\gamma}{2} - W(t) e^{r(T-t)} \right], \tag{3.1}$$

$$u_d^*(t) = \frac{1}{2\rho} \cdot \frac{A}{(\mu - r)} e^{(A-r)(T-t)}, \tag{3.2}$$

$$u_c^*(t) = \frac{1}{2\rho} \cdot \frac{A}{(\mu - r)} e^{-r(T-t)}, \tag{3.3}$$

$$u_{cd}^*(t) = \theta(t) \cdot W(t), \tag{3.4}$$

where A is defined in (2.3), and $\theta(t)$ in (3.4) is given by the unique solution to the following integral equation:

$$\theta(t) = \frac{A}{\rho(\mu - r)} \left\{ e^{-\int_t^T \left(r + (\mu - r)\theta(\tau) - \sigma^2\theta^2(\tau)\right) d\tau} + \rho e^{-\int_t^T \sigma^2\theta^2(\tau) d\tau} - \rho \right\}. \tag{3.5}$$

Proof. See Basak and Chabakauri (2010); Pedersen and Peskir (2017); Zhou and Li (2000); and Bjork, Murgoci, and Zhou (2014). The existence and uniqueness of the solution to the integral equation (3.5) is established in Bjork, Murgoci, and Zhou (2014). ∎

Including the CP strategy (2.12) in this discussion would therefore result in five different investment strategies under consideration. However, Lemma 3.2 shows that there are only three fundamentally different forms of the resulting controls: (i) The DOMV- and cTCMV-optimal controls ((3.2) and (3.3), respectively) are simply deterministic functions of time and do not depend on the investor's wealth. (ii) Both the CP strategy (2.12) and the dTCMV-optimal strategy (3.4) are proportional strategies, in that they specify the amount to invest in the risky asset as a proportion of the wealth at time t. In contrast to the constant proportion θ_{cp} used by the CP strategy, the dTCMV strategy specifies a proportion $\theta(t)$ that is a deterministic function of time satisfying (3.5). (iii) The PCMV-optimal control (3.1) can be viewed as a linear combination of the TCMV-optimal control (3.3) and the CP strategy (2.12).

Starting from a given initial wealth $w_0 > 0$ at time $t_0 \equiv 0$, we now assume that the optimal investment strategies from Lemma 3.2, as well as the CP strategy (2.12), are implemented over the investment time horizon $[t_0, T]$. As a result, we obtain the optimal terminal wealth $W_j(T)$ corresponding to each investment strategy $j \in \{p, d, c, cd\}$, as well as the terminal wealth under the CP strategy $W_{cp}(T)$.

Lemma 3.3 (optimal terminal wealth). *Let $w_0 > 0$ and $t_0 = 0$. Under Assumption 3.1, the optimal terminal wealth $W_j(T)$ corresponding to each investment strategy $j \in \{p, d, c, cd\}$,* *given controlled wealth dynamics* (2.1) *and optimal controls as in Lemma* 3.2*, are given by*

$$W_p(T) = \frac{\gamma}{2} - \left[\frac{\gamma}{2} - w_0 e^{rT}\right] \exp\left\{-\frac{3}{2}AT - \sqrt{A}\cdot Z(T)\right\}, \tag{3.6}$$

$$W_d(T) = w_0 e^{rT} - \frac{1}{2\rho}\left(1 - e^{-AT}\right) + \frac{1}{2\rho}\sqrt{A}\int_0^T e^{A(T-t)} dZ(t), \tag{3.7}$$

$$W_c(T) = w_0 e^{rT} + \frac{1}{2\rho}AT + \frac{1}{2\rho}\sqrt{A}\cdot Z(T), \tag{3.8}$$

$$W_{cd}(T) = w_0 e^{rT}\cdot \exp\left\{\int_0^T \left[(\mu - r)\theta(t) - \frac{1}{2}\sigma^2\theta^2(t)\right] dt + \int_0^T \sigma\theta(t)\, dZ(t)\right\}. \tag{3.9}$$

The terminal wealth $W_{cp}(T)$ under a CP strategy $u^*_{cp}(t) = \theta_{cp} W(t)$ is given by

$$W_{cp}(T) = w_0 e^{rT}\cdot \exp\left\{\left[(\mu - r)\theta_{cp} - \frac{1}{2}\sigma^2\theta_{cp}^2\right] T + \sigma\theta_{cp} Z(T)\right\}. \tag{3.10}$$

Proof. The result (3.6), reported in Vigna (2014) and Pedersen and Peskir (2017), can be obtained by applying Itô's lemma to the auxiliary process

$$X_p(t) = \frac{\gamma}{2}e^{-r(T-t)} - W_p(t), \qquad t \in (t_0 = 0, T], \tag{3.11}$$
$$X_p(t_0) = \frac{\gamma}{2}e^{-rT} - w_0,$$

which shows that $X_p(t)$ follows a GBM (Vigna (2014)). The proof of (3.7)–(3.10) is straightforward and therefore omitted. ■

Based on the results of Lemma 3.3, the distribution of terminal wealth can be identified easily in all cases except for the PCMV-optimal terminal wealth $W_p(T)$, as the following lemma confirms.

Lemma 3.4 (Distribution of terminal wealth under the DOMV, cTCMV, dTCMV, and CP strategies). *Under Assumption* 3.1*, the terminal wealth under the optimal controls of problems DOMV and cTCMV is normally distributed. Specifically, $W_d(T) \sim N\left(\hat{\mu}_d, \hat{\sigma}_d^2\right)$, where*

$$\hat{\mu}_d := E_{u_d^*}^{w_0, t_0=0}\left[W_d(T)\right] = w_0 e^{rT} + \frac{1}{2\rho}\left(e^{AT} - 1\right), \tag{3.12}$$

$$\hat{\sigma}_d^2 := Var_{u_d^*}^{w_0, t_0=0}\left[W_d(T)\right] = \frac{1}{2}\left(\frac{1}{2\rho}\right)^2\left(e^{2AT} - 1\right), \tag{3.13}$$

while $W_c(T) \sim N\left(\hat{\mu}_c, \hat{\sigma}_c^2\right)$ with

$$\hat{\mu}_c := E_{u_c^*}^{w_0, t_0=0}\left[W_c(T)\right] = w_0 e^{rT} + \frac{1}{2\rho}AT, \tag{3.14}$$

$$\hat{\sigma}_c^2 := Var_{u_c^*}^{w_0, t_0=0}\left[W_c(T)\right] = \left(\frac{1}{2\rho}\right)^2 AT. \tag{3.15}$$

The terminal wealth under the dTCMV-optimal and CP investment strategies is lognormally distributed. In particular, $W_{cd}(T) \sim \mathrm{Logn}\left(\hat{\mu}_{cd}, \hat{\sigma}_{cd}^2\right)$, *where*

$$\hat{\mu}_{cd} := E_{u_{cd}^*}^{w_0,t_0=0}\left[\log W_{cd}(T)\right] = \log w_0 + rT + \int_0^T \left[(\mu - r)\,\theta(t) - \frac{1}{2}\sigma^2\theta^2(t)\right] dt, \tag{3.16}$$

$$\hat{\sigma}_{cd}^2 := Var_{u_{cd}^*}^{w_0,t_0=0}\left[\log W_{cd}(T)\right] = \int_0^T \sigma^2\theta^2(t)\, dt, \tag{3.17}$$

while $W_{cp}(T) \sim \mathrm{Logn}\left(\hat{\mu}_{cp}, \hat{\sigma}_{cp}^2\right)$ *with*

$$\hat{\mu}_{cp} := E_{u_{cp}^*}^{w_0,t_0=0}\left[\log W_{cp}(T)\right] = \log w_0 + rT + \left[(\mu - r)\,\theta_{cp} - \frac{1}{2}\sigma^2\theta_{cp}^2\right] T, \tag{3.18}$$

$$\hat{\sigma}_{cp}^2 := Var_{u_{cp}^*}^{w_0,t_0=0}\left[\log W_{cp}(T)\right] = \sigma^2\theta_{cp}^2 T. \tag{3.19}$$

Proof. The results follow directly from the results of Lemma 3.3. ∎

It is clear from the results of Lemma 3.3 that the distribution of the PCMV-optimal terminal wealth $W_p(T)$ is significantly more complex than any of the results presented in Lemma 3.4, as it appears not to conform to any of the commonly encountered probability distributions. However, by rearranging (3.6), it is clear that

$$\frac{\frac{\gamma}{2} - W_p(T)}{\frac{\gamma}{2} - w_0 e^{rT}} \sim \mathrm{Logn}\left(\hat{\mu}_p, \hat{\sigma}_p^2\right), \qquad \text{where } \hat{\mu}_p = -\frac{3}{2}AT \text{ and } \hat{\sigma}_p^2 = AT, \tag{3.20}$$

so that the distribution of $W_p(T)$ can perhaps be best described as a "reflected log-normal distribution" (see Goetzmann et al. (2002) where this terminology is used for a random variable with a similar distribution). The following lemma makes use of the observation (3.20) to give the exact distribution of $W_p(T)$.

Lemma 3.5 (Distribution of PCMV-optimal terminal wealth). *Under Assumption 3.1, the cumulative distribution function (CDF) of the terminal wealth under the optimal control of problem PCMV is given by*

$$P_{u_p^*}^{w_0,t_0=0}\left[W_p(T) \le w\right] = \begin{cases} \Phi\left(-\frac{1}{\sqrt{AT}} \cdot \log\left[\frac{\frac{\gamma}{2} - w}{\frac{\gamma}{2} - w_0 e^{rT}}\right] - \frac{3}{2}\sqrt{AT}\right) & \text{if} \quad w < \frac{\gamma}{2}, \\ 1 & \text{otherwise}, \end{cases} \tag{3.21}$$

where $P_{u_p^*}^{w_0,t_0}(\cdot)$ *denotes the probability calculated under the PCMV-optimal control* $u_p^*(t)$ *and given initial wealth* w_0 *at time* t_0, *while* $\Phi(\cdot)$ *denotes the standard normal CDF. Furthermore, the noncentral moments of the PCMV-optimal terminal wealth* $W_p(T)$ *can be expressed as*

$$\begin{aligned} \mathfrak{m}_p^{(n)}(T) &:= E_{u_p^*}^{w_0,t_0=0}\left[W_p^n(T)\right] \\ &= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\left(\frac{\gamma}{2}\right)^{n-k}\left[w_0 e^{rT} - \frac{\gamma}{2}\right]^k \cdot \exp\left\{\frac{1}{2}k\,(k-3)\,AT\right\}, \qquad n \in \mathbb{N}. \end{aligned} \tag{3.22}$$

Proof. The results (3.21) and (3.22) follow from the observation (3.20). With regard to the cases of the CDF (3.21), it should be noted that Vigna (2014) proved that under the stated assumptions (including Assumption 3.1 and dynamics (2.1)), the PCMV-optimal terminal wealth approaches the quadratic target $\frac{\gamma}{2}$ from below, so that it is always the case that $W_p(T) < \frac{\gamma}{2}$. ∎

The first four noncentral moments of the distribution of the PCMV-optimal terminal wealth play an important role in section 4 and are given by the following lemma.

Lemma 3.6 (Distribution of PCMV-optimal terminal wealth: First four noncentral moments). *Under Assumption 3.1, the first four noncentral moments of the distribution of $W_p(T)$ are given by $\mathfrak{m}_p^{(n)}(T) = E_{u_p^*}^{w_0,t_0=0}\left[W_p^n(T)\right]$, $n \in \{1,2,3,4\}$, where*

$$\mathfrak{m}_p^{(1)}(T) = w_0 e^{rT} + e^{-AT}\left(e^{AT}-1\right)\left[\frac{\gamma}{2} - w_0 e^{rT}\right], \tag{3.23}$$

$$\mathfrak{m}_p^{(2)}(T) = \left[\mathfrak{m}_p^{(1)}(T)\right]^2 + e^{-2AT}\left(e^{AT}-1\right)\left[\frac{\gamma}{2} - w_0 e^{rT}\right]^2, \tag{3.24}$$

$$\begin{aligned}\mathfrak{m}_p^{(3)}(T) &= 3\left[\mathfrak{m}_p^{(1)}(T)\right]\left[\mathfrak{m}_p^{(2)}(T)\right] - 2\left[\mathfrak{m}_p^{(1)}(T)\right]^3 \\ &\quad - e^{-3AT}\left[\left(e^{AT}-1\right)^3 + 3\left(e^{AT}-1\right)^2\right]\left[\frac{\gamma}{2} - w_0 e^{rT}\right]^3,\end{aligned} \tag{3.25}$$

$$\begin{aligned}\mathfrak{m}_p^{(4)}(T) &= 4\left[\mathfrak{m}_p^{(1)}(T)\,\mathfrak{m}_p^{(3)}(T)\right] - 6\left[\mathfrak{m}_p^{(1)}(T)\right]^2\left[\mathfrak{m}_p^{(2)}(T)\right] + 3\left[\mathfrak{m}_p^{(1)}(T)\right]^4 \\ &\quad + \left(e^{2AT} - 4e^{-AT} + 6e^{-3AT} - 3e^{-4AT}\right)\left[\frac{\gamma}{2} - w_0 e^{rT}\right]^4.\end{aligned} \tag{3.26}$$

Proof. The results follow from Lemma 3.5, where the moments (3.22) are simplified and factorized. ∎

Up to this point, we made no reference to any particular choices made by the investor regarding the risk-aversion parameters $\rho > 0$, embedding parameter $\gamma > 2w_0 e^{rT}$, or CP $\theta_{cp} > 0$. In the next section (section 4), we introduce specific choices for these parameters that, when substituted into the results presented in this section, allow the investor to consider the resulting terminal wealth distributions on a comparable basis.

4. Comparison of terminal wealth distributions.

The analytical results presented in section 3 are used in this section to compare the terminal wealth distributions resulting from implementing the various investment strategies under consideration.

Throughout this discussion, we assume that the investor remains agnostic as to the philosophical perspectives underlying the different approaches to dynamic MV optimization. Specifically, we assume that the investor considers the resulting optimal controls in Lemma 3.2 as well as the CP strategy (2.12) simply as different candidate investment strategies, with each resulting in a terminal wealth distribution that can be assessed according to various prespecified risk and return criteria.

In order to compare the resulting terminal wealth distributions on a fair basis, we introduce the following practical assumption.

Assumption 4.1 (expected value target for terminal wealth). *We assume that, regardless of investment strategy $j \in \{p, d, c, cd, cp\}$, the investor sets a particular target value $\mathcal{E} > w_0 e^{rT}$*

for the expected value of terminal wealth. In other words, the investor requires

$$E^{w_0,t_0=0}_{u^{\mathcal{E}*}_j}\left[W^{\mathcal{E}}_j(T)\right] \equiv \mathcal{E}, \quad \text{with } \mathcal{E} > w_0 e^{rT} \quad \forall j \in \{p,d,c,cd,cp\}, \tag{4.1}$$

where $u^{\mathcal{E}*}_j$ *denotes the optimal control for investment strategy* j *achieving the optimal terminal wealth* $W^{\mathcal{E}}_j(T)$ *with expected value* $\mathcal{E}$. *We will refer to* $W^{\mathcal{E}}_j(T)$ *as the target terminal wealth and to its distribution as the target terminal wealth distribution.*

Using the results of section 3, the targeted expected value (4.1) is achieved as follows. For investment strategies $j \in \{p,d,c,cd\}$, the strategy $u^{\mathcal{E}*}_j$ is found by choosing the appropriate value of γ or ρ in Lemma 3.2, while $u^{\mathcal{E}*}_{cp}$ is found by choosing the appropriate proportion θ_{cp} in (2.12). Specifically, for $j \in \{p,d,c,cd,cp\}$, we respectively set $\gamma \equiv \gamma^{\mathcal{E}}_p$, $\rho \equiv \rho^{\mathcal{E}}_d$, $\rho \equiv \rho^{\mathcal{E}}_c$, $\rho \equiv \rho^{\mathcal{E}}_{cd}$ and $\theta_{cp} \equiv \theta^{\mathcal{E}}_{cp}$, where

$$PCMV\left(\gamma \equiv \gamma^{\mathcal{E}}_p\right): \quad \gamma^{\mathcal{E}}_p = 2w_0 e^{rT} + \frac{2e^{AT}}{\left(e^{AT}-1\right)}\left(\mathcal{E} - w_0 e^{rT}\right), \tag{4.2}$$

$$DOMV\left(\rho \equiv \rho^{\mathcal{E}}_d\right): \quad \rho^{\mathcal{E}}_d = \frac{\left(e^{AT}-1\right)}{2\left(\mathcal{E} - w_0 e^{rT}\right)}, \tag{4.3}$$

$$cTCMV\left(\rho \equiv \rho^{\mathcal{E}}_c\right): \quad \rho^{\mathcal{E}}_c = \frac{AT}{2\left(\mathcal{E} - w_0 e^{rT}\right)}, \tag{4.4}$$

$$\begin{aligned} dTCMV\left(\rho \equiv \rho^{\mathcal{E}}_{cd}\right): \quad & \rho^{\mathcal{E}}_{cd} \text{ together with the function } t \to \theta^{\mathcal{E}}(t) \text{ determined numerically} \\ & \text{using (3.5) such that } E^{w_0,t_0=0}_{u^{\mathcal{E}*}_{cd}}\left[W^{\mathcal{E}}_{cd}(T)\right] \equiv \mathcal{E}, \end{aligned} \tag{4.5}$$

$$CP\left(\theta_{cp} \equiv \theta^{\mathcal{E}}_{cp}\right): \quad \theta^{\mathcal{E}}_{cp} = \frac{\log\left(\mathcal{E}/w_0\right) - rT}{(\mu - r)\cdot T}. \tag{4.6}$$

Using the results of Lemmas 3.4 and 3.6, it is straightforward to verify that the choices (4.2)–(4.6) result in the terminal wealth distributions with the required expected value target $\mathcal{E}$.

Remark 4.1 (risk preferences and the basis for comparing terminal wealth distributions). Assumption 4.1 is clearly reasonable from the classical Markowitz (1952) perspective, where, according to one interpretation, the investor simply wishes to achieve the lowest variance for a *given* expected value (see, for example, Perrin and Roncalli (2020)). It is therefore not surprising that when different investment strategies are compared in the literature, it is often on the basis of a fixed level/target of either the expected value or, alternatively, of the volatility of portfolio wealth or returns. For some recent examples, see Bender, Blackburn, and Sun (2019); Dopfel and Lester (2018); Soupé, Lu, and Leote de Carvalho (2019); Zhang, Zohren, and Roberts (2020). According to this view, the scalarization or risk-aversion parameter ρ in (2.4) would be "calibrated" (Bender, Blackburn, and Sun (2019)) on the basis of the chosen target, which in our case results in the particular values (4.2)–(4.5). This sidesteps the explicit selection of a value of ρ appropriate for the investor, a matter on which the literature offers very little guidance (Vigna (2014)), and it also avoids the selection of some arbitrary value of ρ to be used for illustrative purposes without any reference to the investor's goals (as is commonly used in the literature to illustrate analytical results; see, for example, DeMiguel et al. (2020)).

A possible objection to this perspective and therefore to Assumption 4.1, is that using different parameters (4.2)–(4.5) implies that we are comparing the results of different MV problem formulations on the basis of different levels of risk aversion, since different values of ρ are effectively being used in (2.4).

Suppose, for the sake of argument, that we intend to compare the terminal wealth distributions corresponding to the same value of the risk-aversion parameter ρ in (2.4) for all formulations of the MV problem. First, to the detriment of the subsequent results, we will have to exclude the CP strategy from the comparison, since its definition (2.12) does not explicitly incorporate any notion of a risk-aversion parameter, and therefore it is not clear how to select θ_{cp} to ensure a fair comparison on the basis of risk preferences. Next, in the case of the dTCMV problem (2.10), from the perspective of $t_0 \equiv 0$ the effective risk-aversion parameter at time $t \in (0,T]$ depends on the wealth at (future) time t and is therefore stochastic (see Bensoussan, Wong, and Yam (2019) and Bjork, Murgoci, and Zhou (2014) for a detailed analysis).

This leaves the PCMV, DOMV, and cTCMV problems. We observe that by Remark 2.1, the value of $\gamma_p^{\mathcal{E}}$ in (4.2) is consistent with a scalarization parameter value $\rho_p^{\mathcal{E}}$ in the original MV objective (2.4) given by

$$
\begin{aligned}
\rho_p^{\mathcal{E}} &= \frac{e^{AT}-1}{2\left(\mathcal{E}-w_0e^{rT}\right)} \qquad &&(\text{by } (2.6) \text{ and } (4.2))\,, \\
&= \rho_d^{\mathcal{E}} &&(\text{by } (4.3))\,.
\end{aligned}
\tag{4.7}
$$

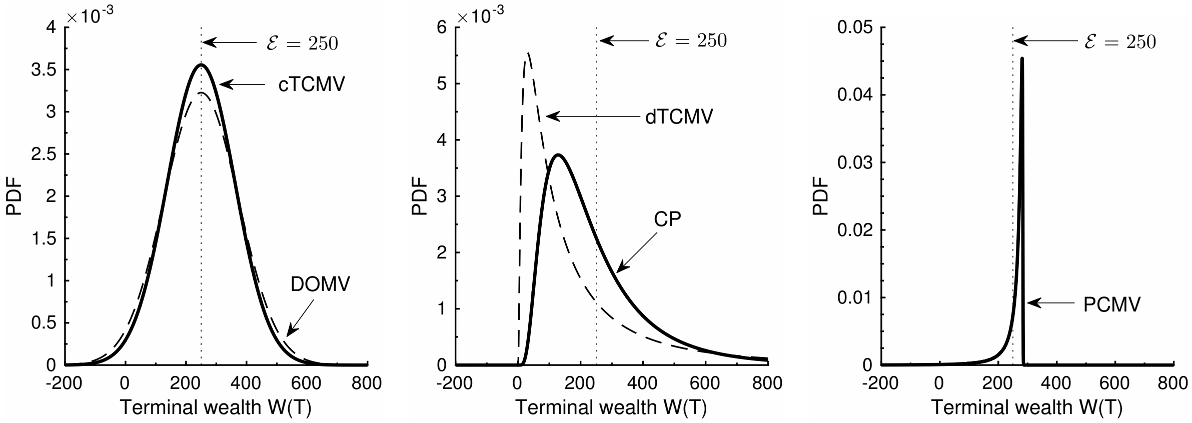
From the perspective of the MV objective (2.4), the PCMV and DOMV problems with the same expected terminal wealth $\mathcal{E}$ therefore make use of identical risk-aversion parameter values. However, this does *not* mean that the PCMV problem with $\gamma \equiv \gamma_p^{\mathcal{E}}$ (4.2) and the DOMV problem with $\rho \equiv \rho_d^{\mathcal{E}}$ (4.3) incorporate the same investor risk preferences for $t \in (0,T]$. Instead, (4.7) only implies that PCMV and DOMV risk preferences agree instantaneously at $t_0 \equiv 0$ (Vigna (2020)).

It is worth emphasizing that the issues involved are subtle and beyond the scope of this paper. Vigna (2017, 2020) rigorously defines and analyzes the notion of "preferences consistency" in dynamic MV optimization approaches, which can informally be defined as the case when the investor's risk preferences at time $t \in (0,T]$ agree with the investor's original risk preferences at time $t_0 \equiv 0$. Vigna (2017, 2020) show that only the DOMV approach is "preferences-consistent," i.e., instantaneously consistent with the investor's original risk preferences at any time $t \in (0,T]$. The PCMV approach is consistent with the target $\gamma/2$ but not with initial risk preferences (Cong and Oosterlee (2016a)). In addition, Vigna (2020) shows that the cTCMV investor is also not preferences-consistent, which is to be expected, since as shown originally in Bjork and Murgoci (2010), the TCMV problem is equivalent to a stochastic control problem with a different objective but no time-consistency constraint, namely the mean-quadratic variation problem (see Van Staden, Dang, and Forsyth (2019) for a detailed analysis).

Therefore, insisting that the resulting terminal wealth distributions should be compared on the basis of equal risk preferences is not only less practical than setting a risk or return target as in Assumption 4.1 but also arguably meaningless in the context of dynamic MV-optimal investment strategies.

Figure 4.1 illustrates the probability density functions (PDFs) of the distributions of $W_j^{\mathcal{E}}(T), j \in \{p, d, c, cd, cp\}$, for the particular choices (4.2)–(4.6), all with the same expected value $\mathcal{E} = 250$. In the case of $j \in \{d, c, cd, cp\}$, these PDFs can be obtained analytically by appropriately substituting (4.3)–(4.6) into the corresponding results of Lemma 3.4. In the case of PCMV ($j = p$), the simulated PDF of $W_p^{\mathcal{E}}(T)$ can be obtained using the expression (3.6) in Lemma 3.3 with $\gamma = \gamma_p^{\mathcal{E}}$ as per (4.2).

The rest of this section is devoted to a quantitative analysis of the differences in the distributions of $W_j^{\mathcal{E}}(T)$ for investment strategies $j \in \{p, d, c, cd, cp\}$, illustrated by Figure 4.1.



(a) cTCMV vs. DOMV (b) CP vs. dTCMV (c) PCMV

Figure 4.1. *Probability density functions (PDFs) of the target terminal wealth $W_j^{\mathcal{E}}(T)$ for $j \in \{p, d, c, cd, cp\}$, all with the same expected value $\mathcal{E} = 250$. $w_0 = 100$, $t_0 = 0$, $T = 10$, and other parameters are as in section 5, so that $w_0 e^{rT} = 106.43$. Note that the same scale is used on the x-axis.*

As an introductory result, the following lemma gives a relationship between the parameters of the target terminal wealth distributions in the case of the CP and dTCMV strategies that turns out to have far-reaching consequences.

Lemma 4.2 (parameters of the distribution of $W_j^{\mathcal{E}}(T)$, $j \in \{cd, cp\}$: CP vs. dTCMV). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. For any target value $\mathcal{E}$ satisfying (4.1), the parameters $\hat{\mu}_j^{\mathcal{E}}$ and $\hat{\sigma}_j^{\mathcal{E}}$ of the lognormally distributed target terminal wealth distributions, $W_j^{\mathcal{E}}(T) \sim \mathrm{Logn}(\hat{\mu}_j^{\mathcal{E}}, (\hat{\sigma}_j^{\mathcal{E}})^2)$, $j \in \{cp, cd\}$, satisfy the following relationships:*

$$\hat{\mu}_{cp}^{\mathcal{E}} \geq \hat{\mu}_{cd}^{\mathcal{E}}, \qquad \hat{\sigma}_{cp}^{\mathcal{E}} \leq \hat{\sigma}_{cd}^{\mathcal{E}}. \tag{4.8}$$

Proof. By Lemma 3.4, $\hat{\mu}_{cp}^{\mathcal{E}} = \log(\mathcal{E}) - \frac{1}{2}\left(\hat{\sigma}_{cp}^{\mathcal{E}}\right)^2$ and $\hat{\mu}_{cd}^{\mathcal{E}} = \log(\mathcal{E}) - \frac{1}{2}(\hat{\sigma}_{cd}^{\mathcal{E}})^2$, so we only need to prove that $\hat{\sigma}_{cp}^{\mathcal{E}} \leq \hat{\sigma}_{cd}^{\mathcal{E}}$, where

$$\hat{\sigma}_{cp}^{\mathcal{E}} = \frac{1}{\sqrt{AT}}\left[\log(\mathcal{E}/w_0) - rT\right], \qquad \hat{\sigma}_{cd}^{\mathcal{E}} = \sigma \cdot \left(\int_0^T \left[\theta^{\mathcal{E}}(t)\right]^2 dt\right)^{\frac{1}{2}}. \tag{4.9}$$

To ensure that $W_{cd}^{\mathcal{E}}(T)$ has the required mean $\mathcal{E}$, the function $t \to \theta^{\mathcal{E}}(t)$ and risk-aversion parameter $\rho_{cd}^{\mathcal{E}}$ in (4.5) are solved numerically using the integral equation (3.5) to guarantee that

$$\int_0^T \theta^{\mathcal{E}}(t)\,dt \equiv \frac{\log(\mathcal{E}/w_0) - rT}{(\mu - r)}. \tag{4.10}$$

With $\theta_{cp}^{\mathcal{E}}$ defined as the CP in (4.6), we recognize that $\theta_{cp}^{\mathcal{E}} T = \int_0^T \theta^{\mathcal{E}}(t)\,dt$. Furthermore, the Cauchy–Schwarz inequality implies that

$$\frac{1}{T}\left(\theta_{cp}^{\mathcal{E}} T\right)^2 = \frac{1}{T}\left(\int_0^T \theta^{\mathcal{E}}(t)\,dt\right)^2 \leq \int_0^T \left[\theta^{\mathcal{E}}(t)\right]^2 dt. \tag{4.11}$$

Therefore, (4.9) and (4.11) imply that we always have $\hat{\sigma}_{cp}^{\mathcal{E}} \leq \hat{\sigma}_{cd}^{\mathcal{E}}$, regardless of the target $\mathcal{E} > w_0 e^{rT}$. ■

As noted before, the dTCMV-optimal strategy is an example of a deterministic "glide path" strategy typically encountered in the pension fund literature, and in that particular context the result (4.11) used in the proof of Lemma 4.2 is a known result (see, for example, Forsyth and Vetzal (2019a); Graf (2017)). However, it is worth emphasizing the result (4.8) in this paper for two reasons.

First, in the specific case of the dTCMV problem, the conclusion of Lemma 4.2 enables the comparison of the distributions of $W_{cd}^{\mathcal{E}}(T)$ and $W_{cp}^{\mathcal{E}}(T)$ without resorting to the numerical solution of the function $t \to \theta^{\mathcal{E}}(t)$ using the cumbersome integral equation (3.5). In particular, note that the exact form of the function $t \to \theta^{\mathcal{E}}(t)$ does not matter; the only relevant fact regarding $\theta^{\mathcal{E}}(t)$ is that its integral satisfies (4.10), which is just a constant multiple of the value of $\theta_{cp}^{\mathcal{E}}$ in (4.6). Second, the result (4.8) turns out to be sufficient to prove a number of very interesting results not just limited to mean and variance but also including a first-order stochastic dominance result (see Theorem 4.13 below). This follows since we have a complete description of the relevant distributions under the stated assumptions.

We now return to our comparison of the distributions of the target terminal wealth $W_j^{\mathcal{E}}(T)$ for investment strategies $j \in \{p, d, c, cd, cp\}$. First, consider an investor primarily interested in the first two moments of the terminal wealth. Since all the target terminal wealth distributions $W_j^{\mathcal{E}}(T)$ have the same mean $\mathcal{E}$ as per (4.1), we start by considering the variance $W_j^{\mathcal{E}}(T)$ obtained for each investment strategy j.

Lemma 4.3 (Variance: Target terminal wealth distribution). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. The variance of the target terminal wealth $W_j^{\mathcal{E}}(T)$, for $j \in \{p, d, c, cd, cp\}$, is given by the following expressions:*

$$Var_{u_p^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_p^{\mathcal{E}}(T)\right] = \frac{1}{(e^{AT}-1)}\left(\mathcal{E} - w_0 e^{rT}\right)^2, \quad Var_{u_d^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_d^{\mathcal{E}}(T)\right] = \frac{\left(e^{AT}+1\right)}{2\left(e^{AT}-1\right)}\left(\mathcal{E} - w_0 e^{rT}\right)^2, \tag{4.12}$$

$$Var_{u_c^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_c^{\mathcal{E}}(T)\right] = \frac{1}{AT}\left(\mathcal{E} - w_0 e^{rT}\right)^2, \quad Var_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_j^{\mathcal{E}}(T)\right] = \mathcal{E}^2 \cdot \left(e^{\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2} - 1\right), j \in \{cd, cp\}, \tag{4.13}$$

where $\hat{\sigma}_j^{\mathcal{E}}, j \in \{cp, cd\}$ are given by (4.9).

Proof. The results follow from Lemma 3.4, Lemma 3.6, and (4.2)–(4.6). ■

The following lemma compares the variances of the target terminal wealth distributions.

Lemma 4.4 (Comparison: Variance). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. The variances of the target wealth distributions for investment strategies $j \in \{p, d, c, cd, cp\}$ are related as follows:*

$$Var_{u_p^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_p^{\mathcal{E}}(T)\right] < Var_{u_c^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_c^{\mathcal{E}}(T)\right] \tag{4.14}$$

$$< \begin{cases} Var_{u_d^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_d^{\mathcal{E}}(T)\right], \\ Var_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cp}^{\mathcal{E}}(T)\right] \leq Var_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cd}^{\mathcal{E}}(T)\right]. \end{cases} \tag{4.15}$$

Proof. Inequality (4.14) is obvious from the variance results (4.12)–(4.13) in Lemma 4.3. Considering (4.15), we first observe that $(x-2)\,e^x + x + 2 > 0$ for all $x > 0$. Since $A > 0$ (recall that $\mu > r$, $\sigma > 0$) and $T > 0$, $AT > 0$, we exploit the following inequality which turns out to be very useful for proving some of the subsequent results:

$$AT > \frac{2\left(e^{AT}-1\right)}{\left(e^{AT}+1\right)} \quad \forall A, T > 0. \tag{4.16}$$

Considering the results of Lemma 4.3, the inequality (4.16) implies that $Var_{u_c^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_c^{\mathcal{E}}(T)\right] < Var_{u_d^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_d^{\mathcal{E}}(T)\right]$. Next, observing that $\log x \geq [1-(1/x)]$ for all $x > 1$, and $\exp\left\{y \cdot \log^2 x\right\} > \left[1 + y \cdot \log^2 x\right]$ for all $x, y > 0$, it follows that

$$\exp\left\{y \cdot \log^2 x\right\} - y\left(1 - \frac{1}{x}\right)^2 - 1 > 0 \quad \forall x > 1, y > 0. \tag{4.17}$$

Since $\mathcal{E}/\left(w_0 e^{rT}\right) > 1$ by (4.1) and $AT > 0$, (4.17) implies that we also have $Var_{u_c^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_c^{\mathcal{E}}(T)\right] < Var_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cp}^{\mathcal{E}}(T)\right]$. Finally, the conclusion $Var_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cp}^{\mathcal{E}}(T)\right] \leq Var_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cd}^{\mathcal{E}}(T)\right]$ follows from (4.13) and (4.8). ■

Lemma 4.4 therefore shows that a hypothetical MV investor who is only narrowly interested in the mean and variance of terminal wealth and agnostic as to the philosophical differences underlying the various approaches to dynamic MV optimization would conclude the following: (i) The PCMV strategy always outperforms all the other strategies; (ii) the cTCMV strategy outperforms both the DOMV and CP strategies; and (iii) as expected based on the result of Lemma 4.2, the CP strategy outperforms the dTCMV strategy. Our analytical results therefore confirm and assist in explaining the conclusions from numerical tests regarding the relative performance of the PCMV and the CP strategies in Forsyth and Vetzal (2017b), as well as the performance comparison of the PCMV, cTCMV, dTCMV, and CP strategies presented in Forsyth and Vetzal (2019a).

Remark 4.5 (comparison of quantities other than mean and variance). The subsequent results include the comparison of higher-order moments, median values, cumulative distribution functions, and downside risk measures associated with the target terminal wealth distributions obtained under the various MV approaches. However, since the investor is performing MV optimization, a question might arise as to why aspects of the distribution other than mean and variance might be of importance to the investor. Furthermore, if other qualities of the distribution are important, should these be incorporated in the objective function?

First, as observed in the introduction, dynamic MV optimization appears to be very popular in institutional settings. Some recent applications include deriving optimal investment strategies for pension funds (see, for example, Forsyth and Vetzal (2019a); Forsyth, Vetzal, and Westmacott (2019); Højgaard and Vigna (2007); Liang, Bai, and Guo (2014); Menoncin and Vigna (2013); Nkeki (2014); Sun, Li, and Zeng (2016); Vigna (2014); Wang and Chen (2018, 2019); Wu and Zeng (2015)), solving investment-reinsurance problems faced by insurance providers (Bi and Cai (2019); Chen, Li, and Guo (2013); Li and Li (2013); Lin and Qian (2016); Zhao, Shen, and Zeng (2016); Zhou et al. (2016)), optimization in corporate international investment (Long and Zeng (2016)), and asset-liability management (Peng, Cui, and Shi (2018); Wei and Wang (2017); Zhang et al. (2017); Zweng and Li (2011)). In all of these practical settings, it is highly likely that the investor and other stakeholders will be concerned with other aspects of the distribution in addition to its mean and variance. Not only might the investor have secondary risk and investment performance considerations (for example, other risk and return measures might have to be reported even though they are not explicitly included in the optimization), but external stakeholders such as regulators might require the investor to consider other aspects of the distribution (see, for example, Antolin et al. (2009)), including downside risk measures, such as expected shortfall and VaR which are discussed below.

Of course, the investor might wish to augment the objective function to include aspects of the distribution other than mean and variance. Back, Crane, and Crotty (2018) observe that there is evidence indicating that investors are concerned with higher-order moments, and portfolio optimization with higher-order moments has in fact been proposed (see, for example, Aracioglu, Demircan, and Soyuer (2011); Jondeau and Rockinger (2006); Jurczenko, Maillet, and Merlin (2012); Lai, Yu, and Wang (2006); Maringer and Parpas (2009)). Furthermore, if downside risk is a major consideration, the investor might replace variance in the objective with a downside risk measure (see, for example, Forsyth (2020); Miller and Yang (2017)).

However, as the MV objective remains by far the most popular objective function in the recent dynamic portfolio optimization literature, and (as noted above) is especially popular in applications in institutional settings, we correspondingly focus on comparing the terminal wealth distributions in the case of MV optimization, leaving other formulations for our future work.

In the next two lemmas, we focus on the skewness and (excess) kurtosis of the target wealth distribution, since these are the quantities typically included in portfolio optimization problems that generalize MV optimization to include higher-order moments; see, for example, Jurczenko, Maillet, and Merlin (2012). We remind the reader, that as discussed in Goetzmann et al. (2002), dynamic trading strategies essentially contain embedded options. Hence it is useful to compare the higher moments of the various strategies.

Lemma 4.6 compares the skewness[3] of the target terminal wealth distributions.

Lemma 4.6 (Comparison: Skewness). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. The coefficients of skewness of the target wealth distributions,*

$$Skew_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}[W_j^{\mathcal{E}}(T)], \quad j \in \{p, d, c, cd, cp\},$$

are related as follows:

$$\begin{aligned} Skew_{u_p^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_p^{\mathcal{E}}(T)\right] < 0 &= Skew_{u_c^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_c^{\mathcal{E}}(T)\right] \\ &= Skew_{u_d^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_d^{\mathcal{E}}(T)\right] \qquad (4.18) \\ &< Skew_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cp}^{\mathcal{E}}(T)\right] \qquad (4.19) \\ &\leq Skew_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cd}^{\mathcal{E}}(T)\right]. \qquad (4.20) \end{aligned}$$

Proof. From Lemma 3.6, it follows that

$$Skew_{u_p^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_p^{\mathcal{E}}(T)\right] = -\left(e^{AT}-1\right)^{\frac{1}{2}}\left[\left(e^{AT}-1\right)+3\right] < 0 \quad \forall A, T > 0, \qquad (4.21)$$

which, together with Lemma 3.4, implies (4.18). It follows from Lemma 4.2 that

$$Skew_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_j^{\mathcal{E}}(T)\right] = \left[e^{\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2}+2\right]\cdot\left[e^{\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2}-1\right]^{\frac{1}{2}}, \; j \in \{cd, cp\}, \qquad (4.22)$$

which implies (4.19) and, together with (4.8), also implies (4.20). ■

Before discussing the implications of Lemma 4.6, we present the comparison of the excess kurtosis of the target terminal wealth distributions.

Lemma 4.7 (Comparison: Excess kurtosis). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. The coefficients of (excess) kurtosis of the target wealth distributions, $Kurt_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}[W_j^{\mathcal{E}}(T)]$, $j \in \{p, d, c, cd, cp\}$, are related as follows:*

$$0 = Kurt_{u_c^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_c^{\mathcal{E}}(T)\right] = Kurt_{u_d^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_d^{\mathcal{E}}(T)\right] \qquad (4.23)$$

$$< \begin{cases} Kurt_{u_p^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_p^{\mathcal{E}}(T)\right], \\ Kurt_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cp}^{\mathcal{E}}(T)\right] \leq Kurt_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cd}^{\mathcal{E}}(T)\right]. \end{cases} \qquad (4.24)$$

Proof. Equation (4.23) follows from Lemma 3.4. Noting the factorization

$$\begin{aligned} &e^{2AT} - 4e^{-AT} + 6e^{-3AT} - 3e^{-4AT} \\ &= e^{-4AT}\left(e^{AT}-1\right)^2\left[\left(e^{AT}-1\right)^4 + 6\left(e^{AT}-1\right)^3 + 15\left(e^{AT}-1\right)^2 + 16\left(e^{AT}-1\right) + 3\right], \end{aligned}$$

[3] We use the standard definition of Pearson's moment coefficient of skewness, which in this context is simply given by $Skew_{u_j^{\mathcal{E}*}}^{w_0,t_0}\left[W_j^{\mathcal{E}}(T)\right] = E_{u_j^{\mathcal{E}*}}^{w_0,t_0}[(W_j^{\mathcal{E}}(T)-\mathcal{E})^3]/[Var_{u_j^{\mathcal{E}*}}^{w_0,t_0}[W_j^{\mathcal{E}}(T)]]^{3/2}$.

we see that Lemma 3.6 implies that the excess kurtosis of $W_p^{\mathcal{E}}(T)$ is always positive,

$$Kurt_{u_p^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_p^{\mathcal{E}}(T)\right] = \left(e^{AT}-1\right)\left[\left(e^{AT}-1\right)^3 + 6\left(e^{AT}-1\right)^2 + 15\left(e^{AT}-1\right) + 16\right] > 0. \tag{4.25}$$

In the case of CP and dTCMV, Lemma 4.2 implies that

$$Kurt_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_j^{\mathcal{E}}(T)\right] = e^{4\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2} + 2e^{3\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2} + 3e^{2\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2} - 6 > 0, \quad j \in \{cd, cp\}, \tag{4.26}$$

which, together with (4.8), implies (4.24). ■

Considering the results of Lemmas 4.6 and 4.7, we note that there is overwhelming evidence in the literature that investors prefer positive skewness under very general assumptions; see, for example, Agren (2006); Back, Crane, and Crotty (2018); Barberis and Huang (2008); Barberis, Harvey, and Shephard (2016); Boyer, Mitton, and Vorkink (2010); Goetzmann and Kumar (2008); Hagestande and Wittussen (2016); Heuson, Hutchinson, and Kumar (2020); Kumar (2009); Maringer and Parpas (2009); Mitton and Vorkink (2007); Omed and Song (2014), among many others. This appears to follow from an investor preference for the possibility of a large gain (Agren (2006)), which may not be entirely rational (Omed and Song (2014)). In contrast, the evidence on kurtosis preferences is far more complicated;[4] see, for example, Haas (2007). However, when portfolio optimization with higher-order moments is performed (see, for example, Jurczenko, Maillet, and Merlin (2012)), kurtosis is usually minimized, suggesting that lower kurtosis is preferred (Maringer and Parpas (2009)).

Based on these observations, the results of Lemmas 4.6 and 4.7 indicate that the excess kurtosis and, especially, the negative skewness associated with the PCMV-optimal strategy are at least somewhat undesirable from the perspective of an investor concerned with higher-order moments. The desirable variance result reported in Lemma 4.4 for the PCMV strategy therefore comes at the cost of other potentially undesirable shape characteristics. These results therefore explain the numerical results reported in Forsyth and Vetzal (2019a), where the increased left tail risk of the PCMV strategy compared to the cTCMV and CP strategies is observed.

We also observe that the dTCMV strategy not only results in the largest (positive) skewness but also is associated with the largest variance and the largest excess kurtosis. The normally distributed terminal wealth of the DOMV and cTCMV strategies results in zero skewness and excess kurtosis, as expected. Therefore, for an investor concerned with the first four moments, the cTCMV strategy is always to be preferred to the DOMV strategy, since the associated target terminal wealth distributions have the same mean (Assumption 4.1) and the same skewness and kurtosis (Lemmas 4.6 and 4.7), but the cTCMV strategy has a lower variance (Lemma 4.4).

Finally, we note the interesting fact that the skewness and kurtosis results for the CP and dTCMV strategies depends on the target $\mathcal{E}$, but this is not the case for the PCMV, cTCMV, and DOMV strategies. As discussed in section 5, this has some interesting consequences.

[4]As Haas (2007) notes, "while risk aversion implies that investors dislike large losses more than they like large profits, kurtosis aversion requires that they dislike fat tails more than they like high peaks."

Given the preceding results on skewness and kurtosis, and the fact that as per Assumption 4.1 all the target distributions considered in this section have identical means $\mathcal{E}$, the comparison of the median terminal wealth outcomes, given in the following lemma, is instructive. All else being equal, investors are expected to prefer larger median values (Forsyth, Vetzal, and Westmacott (2019)).

Lemma 4.8 (Comparison: Medians). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. The medians of the target wealth distributions,*

$$Med_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_j^{\mathcal{E}}(T)\right], \quad j \in \{p, d, c, cd, cp\},$$

are related as follows:

$$\begin{aligned}
&Med_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cd}^{\mathcal{E}}(T)\right] \leq Med_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cp}^{\mathcal{E}}(T)\right] && (4.27)\\
&\qquad < Med_{u_{d}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{d}^{\mathcal{E}}(T)\right] &&\\
&\qquad = Med_{u_{c}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{c}^{\mathcal{E}}(T)\right] && (4.28)\\
&\qquad = \mathcal{E} &&\\
&\qquad < Med_{u_{p}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{p}^{\mathcal{E}}(T)\right]. && (4.29)
\end{aligned}$$

Proof. Since $Med_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}[W_j^{\mathcal{E}}(T)] = \mathcal{E} \cdot \exp\{-\frac{1}{2}(\hat{\sigma}_j^{\mathcal{E}})^2\}$ for $j \in \{cd, cp\}$, results (4.27) and (4.28) follow from Lemmas 3.4 and 4.2. Using Lemma 3.5 and (4.2), it can be shown that

$$Med_{u_p^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_p^{\mathcal{E}}(T)\right] = \mathcal{E} + \left(\frac{1 - e^{-\frac{1}{2}AT}}{e^{AT} - 1}\right)\left(\mathcal{E} - w_0 e^{rT}\right). \tag{4.30}$$

By Assumption 4.1, $\left(\mathcal{E} - w_0 e^{rT}\right) > 0$, so (4.30) implies (4.29). ■

On the basis of median terminal wealth, Lemma 4.8 shows that the investor would prefer the CP strategy to the dTCMV strategy and prefer either the cTCMV or the DOMV strategy to the CP strategy, while the PCMV strategy dominates all other strategies in terms of median wealth. This conclusion therefore provides an analytical explanation of the numerically calculated median results reported in Forsyth and Vetzal (2019a) and Forsyth, Vetzal, and Westmacott (2019).

The following lemma reports the analytical expressions of the cumulative distribution functions (CDFs) of $W_j^{\mathcal{E}}(T)$, for $j \in \{p, d, c, cd, cp\}$.

Lemma 4.9 (CDFs: Target terminal wealth distributions). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. Then the CDFs of the target terminal wealth $W_j^{\mathcal{E}}(T)$,* *for $j \in \{p, d, c, cd, cp\}$, are as follows:*

(4.31)

$$\mathbb{P}^{w_0,t_0=0}_{u_p^{\mathcal{E}*}}\left[W_p^{\mathcal{E}}(T) \leq w\right] = \begin{cases} \Phi\left(-\frac{1}{\sqrt{AT}} \cdot \log\left[1 - \left(\frac{1-e^{-AT}}{\mathcal{E} - w_0 e^{rT}}\right)\left(w - w_0 e^{rT}\right)\right] - \frac{3}{2}\sqrt{AT}\right) & \text{if } w < \left(\frac{\mathcal{E} - w_0 e^{(r-A)T}}{1-e^{-AT}}\right), \\ 1 & \text{otherwise} \end{cases}$$

and

$$\mathbb{P}^{w_0,t_0=0}_{u_d^{\mathcal{E}*}}\left[W_d^{\mathcal{E}}(T) \leq w\right] = \Phi\left(\frac{(w-\mathcal{E})}{\mathcal{E} - w_0 e^{rT}} \cdot \sqrt{\frac{2\left(e^{AT}-1\right)}{\left(e^{AT}+1\right)}}\right), \quad w \in \mathbb{R}, \tag{4.32}$$

$$\mathbb{P}^{w_0,t_0=0}_{u_c^{\mathcal{E}*}}\left[W_c^{\mathcal{E}}(T) \leq w\right] = \Phi\left(\frac{(w-\mathcal{E})}{\left(\mathcal{E} - w_0 e^{rT}\right)} \cdot \sqrt{AT}\right), \quad w \in \mathbb{R}, \tag{4.33}$$

$$\mathbb{P}^{w_0,t_0=0}_{u_j^{\mathcal{E}*}}\left[W_j^{\mathcal{E}}(T) \leq w\right] = \Phi\left(\frac{\log\left(w/\mathcal{E}\right) + \frac{1}{2}\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2}{\hat{\sigma}_j^{\mathcal{E}}}\right), \qquad w > 0,\ j \in \{cd, cp\}, \tag{4.34}$$

where we recall that $\Phi(\cdot)$ denotes the standard normal CDF.

Proof. This follows from the results of Lemmas 3.4 and 3.5, as well as the definitions (4.1) and (4.9). ∎

The remaining results of this section make use of the analytical expressions of the CDFs of $W_j^{\mathcal{E}}(T)$ given in Lemma 4.9. However, considering the results (4.31)–(4.34), it is clear that the distribution of the PCMV-optimal target terminal wealth $W_p^{\mathcal{E}}(T)$ in (4.31) is fundamentally different and far more analytically challenging than the distributions of the target terminal wealth under the other strategies.

We leave further analysis of the PCMV target wealth distribution for our future work, and instead focus on the strategies $j \in \{d, c, cd, cp\}$ in the subsequent analysis. The reason is that in practice it is simply far easier to use (4.31) to numerically calculate and compare desired quantities of interest involving the PCMV target wealth, rather than to derive analytical comparison results which would be significantly more complex and cumbersome to use. By contrast, as we show subsequently, we can derive a number of simple comparison results for strategies $j \in \{d, c, cd, cp\}$, which has very interesting and potentially far-reaching implications for the MV investor.

We now recall the concept of first-order stochastic dominance by applying the definition given in Joshi and Paterson (2013) in our setting.

Definition 4.10 (first-order stochastic dominance). *$W_j^{\mathcal{E}}(T)$ has first-order stochastic dominance over $W_k^{\mathcal{E}}(T)$ for some $j, k \in \{p, d, c, cd, cp\}$ if*

$$\mathbb{P}^{w_0,t_0}_{u_j^{\mathcal{E}*}}\left[W_j^{\mathcal{E}}(T) \leq w\right] \leq \mathbb{P}^{w_0,t_0}_{u_k^{\mathcal{E}*}}\left[W_k^{\mathcal{E}}(T) \leq w\right] \quad \forall w \tag{4.35}$$

and if

$$\mathbb{P}^{w_0,t_0}_{u_j^{\mathcal{E}*}}\left[W_j^{\mathcal{E}}(T) \le w\right] < \mathbb{P}^{w_0,t_0}_{u_k^{\mathcal{E}*}}\left[W_k^{\mathcal{E}}(T) \le w\right] \quad \textit{for some } w. \tag{4.36}$$

We observe that Definition 4.10 is a very general result, since it implies that any investor preferring more wealth to less wealth (i.e., any investor with an increasing utility function) would prefer $W_j^{\mathcal{E}}(T)$ over $W_k^{\mathcal{E}}(T)$ if (4.35)–(4.36) are satisfied.

Remark 4.11 (practical challenges of applying Definition 4.10). While very general, the conditions of Definition 4.10 can be impossible to satisfy in the case of nontrivial investment strategies, including the strategies considered in this paper. In particular, note that (4.35) is required to hold for *all* values of w. Therefore, even when comparing two relatively simple strategies, for example, (i) the CP strategy defined in (2.12), and (ii) the strategy of regularly participating in a lottery with a sufficiently large payout (not conventionally considered an "investment strategy," with good reason), condition (4.35) would be violated despite the fact that strategy (ii) is unlikely to be preferred by any reasonable investor over strategy (i). However, relaxing condition (4.35) by requiring that it holds only for values of w below a certain level is particularly useful, in that it would readily show that strategy (i) is to be preferred over strategy (ii) in this simple example.

As a result of the observations in Remark 4.11, the weaker definition of stochastic dominance proposed by Atkinson (1987) is adapted to our setting and is given by Definition 4.12.

Definition 4.12 (partial first-order stochastic dominance relative to a level ℓ). Let $j,k \in \{p,d,c,cd,cp\}$. We define $W_j^{\mathcal{E}}(T)$ as having partial first-order stochastic dominance over $W_k^{\mathcal{E}}(T)$ relative to a level ℓ if

$$\mathbb{P}^{w_0,t_0}_{u_j^{\mathcal{E}*}}\left[W_j^{\mathcal{E}}(T) \le w\right] \le \mathbb{P}^{w_0,t_0}_{u_k^{\mathcal{E}*}}\left[W_k^{\mathcal{E}}(T) \le w\right] \quad \forall w < \ell. \tag{4.37}$$

Note that Definition 4.12 focuses on "downside risk" in that (4.37) is only concerned with the behavior of the CDFs below the given level ℓ. In what follows, we typically set ℓ equal to the investor's expected value target $\mathcal{E}$. In other words, we assume that the investor is primarily concerned with the possibility of *underperforming* the expected value target while considering the "upside" of outcomes above $\mathcal{E}$ as a satisfying windfall, but this is not critical for investment strategy comparison purposes. We argue that this treatment is reasonable given the popularity of dynamic MV strategies in institutional settings,[5] especially in the case of pension fund managers and insurance companies who are likely to take a keen interest in avoiding the underperformance of expectations.

Using Definition 4.12, the following theorem gives one of the key results of this paper.

Theorem 4.13 (partial first-order stochastic dominance for underperforming expectations). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. We have the following relationships between the CDFs of $W_j^{\mathcal{E}}(T)$ for $j \in \{d,c,cd,cp\}$:*

$$\mathbb{P}^{w_0,t_0=0}_{u_c^{\mathcal{E}*}}\left[W_c^{\mathcal{E}}(T) \le w\right] < \mathbb{P}^{w_0,t_0=0}_{u_d^{\mathcal{E}*}}\left[W_d^{\mathcal{E}}(T) \le w\right] \quad \forall w < \mathcal{E} \tag{4.38}$$

[5]See, for example, Alia, Chighoub, and Sohail (2016); Bi and Cai (2019); Liang, Bai, and Guo (2014); Liang and Song (2015); Lin and Qian (2016); Sun, Li, and Zeng (2016); Vigna (2014); Wu and Zeng (2015), among many others.

and

$$\mathbb{P}^{w_0,t_0=0}_{u^{\mathcal{E}*}_{cp}}\left[W^{\mathcal{E}}_{cp}(T)\leq w\right]\leq\mathbb{P}^{w_0,t_0=0}_{u^{\mathcal{E}*}_{cd}}\left[W^{\mathcal{E}}_{cd}(T)\leq w\right]\quad\forall w<\mathcal{E}. \tag{4.39}$$

Furthermore, there exists a unique value of terminal wealth $w^0_{cp;c}\in(0,\mathcal{E})$*, with the upper bound*

$$w^0_{cp;c}<\frac{\mathcal{E}-w_0e^{rT}}{\log(\mathcal{E}/w_0)-rT}, \tag{4.40}$$

such that

$$\mathbb{P}^{w_0,t_0=0}_{u^{\mathcal{E}*}_{cp}}\left[W^{\mathcal{E}}_{cp}(T)\leq w\right]<\mathbb{P}^{w_0,t_0=0}_{u^{\mathcal{E}*}_{c}}\left[W^{\mathcal{E}}_{c}(T)\leq w\right]\quad\forall w<w^0_{cp;c}, \tag{4.41}$$

$$\mathbb{P}^{w_0,t_0=0}_{u^{\mathcal{E}*}_{c}}\left[W^{\mathcal{E}}_{c}(T)\leq w\right]<\mathbb{P}^{w_0,t_0=0}_{u^{\mathcal{E}*}_{cp}}\left[W^{\mathcal{E}}_{cp}(T)\leq w\right]\quad\forall w\in\left(w^0_{cp;c},\mathcal{E}\right]. \tag{4.42}$$

Proof. Result (4.38) follows from (4.32)–(4.33), the relationship (4.16), and the fact that Φ is strictly increasing. To prove (4.39), we first note that

$$x\log(z)-\frac{1}{2}xy^2-\frac{1}{2}x^2y\leq 0\quad\forall x\geq 0,y\geq 0,z\leq 1.$$

The result (4.39) follows from setting $y=\hat{\sigma}^{\mathcal{E}}_{cp}$, $x=\hat{\sigma}^{\mathcal{E}}_{cd}-\hat{\sigma}^{\mathcal{E}}_{cp}$ (so that $x\geq 0$, by (4.8)), and $z=w/\mathcal{E}$, noting the definition (4.34) and using the fact that Φ is strictly increasing. Next, let $x^0_{cp;c}$ be the unique root in the interval $(0,1)$ of the function $x\to f_{pc;c}(x;c_1,c_2)$, defined by

$$f_{pc;c}(x;c_1,c_2)=\left[\frac{c_1}{c_2}\right]\cdot\log(x)-\left[\frac{c_1e^{c_2}}{e^{c_2}-1}\right]\cdot(x-1)+\frac{1}{2}c_2,\quad x\in(0,1]\ \ (c_1>0,c_2>0).$$

Then (4.40)–(4.42) follows by setting $w^0_{cp;c}=\mathcal{E}\cdot x^0_{cp;c}$, $c_1=AT$, and $c_2=[\log(\mathcal{E}/w_0)-rT]$. ■

The results of Theorem 4.13 are illustrated in Figures 4.2 and 4.3 below and provide theoretical support for the qualitatively similar observations regarding the numerical results[6] presented in Forsyth and Vetzal (2019a). We make the following observations regarding our analytical results.

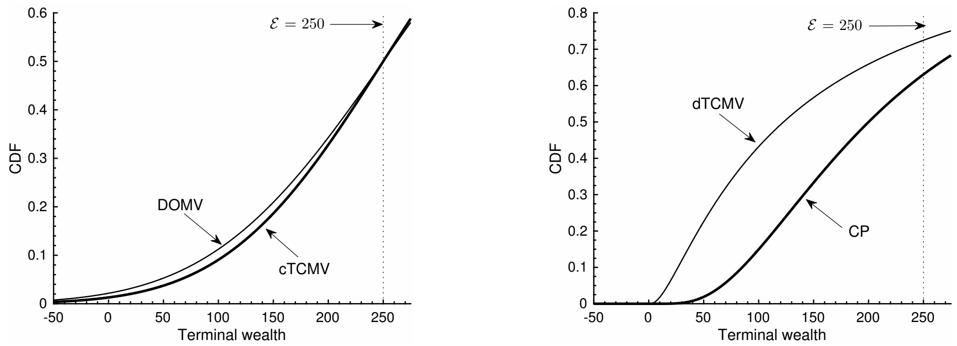
First, subject to the stated assumptions, any investor who is agnostic about the philosophy underlying the different MV optimization approaches and simply concerned about the risk of underperforming the expectation $\mathcal{E}$ would never choose the DOMV or the dTCMV strategy, since better results can be obtained using the cTCMV or the CP strategy, respectively. Note that, as in the case of (4.38), we typically have strict inequality in (4.39) as well, since in typical applications it is the case that $\hat{\sigma}^{\mathcal{E}}_{cd}>\hat{\sigma}^{\mathcal{E}}_{cp}$ in (4.8).

Second, (4.41)–(4.42) indicates that the CP strategy is preferred to the cTCMV strategy if we set the level $\ell\leq w^0_{cp;c}$ in Definition 4.12. Note that the upper bound (4.40) on $w^0_{cp;c}$ is strictly (and often substantially) less than $\mathcal{E}$, so this bound can be very useful for a quick assessment depending on the critical value of w under consideration in (4.41)–(4.42). This behavior is to be expected, since wealth can assume negative values in the case of the cTCMV

[6]The numerical results in Forsyth and Vetzal (2019a) do not include the DOMV-optimal strategy.

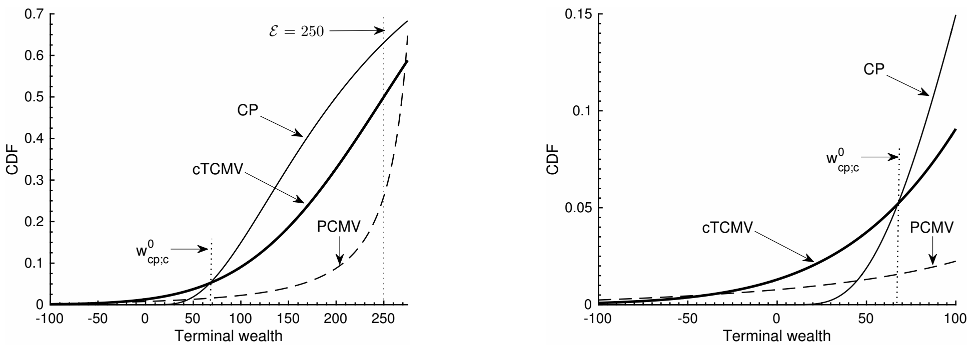
strategy but not in the case of the CP strategy (see Lemma 3.4). However, the skewness results of the target wealth distribution in the case of the CP strategy (see Lemmas 4.6 and 4.8) means that it starts (in aggregate probability) underperforming the cTCMV strategy fairly quickly as $\mathcal{E}$ is approached from below; see Figure 4.3.

For illustrative purposes, Figure 4.3 also includes the simulated CDF of the PCMV target terminal wealth distribution. Compared to the CP and cTCMV strategies, it is clear that the negative skewness (Lemma 4.6) and excess kurtosis (Lemma 4.7) in this case combine to imply that the PCMV-optimal strategy holds substantial downside risks, as noted above.



(a) cTCMV and DOMV (b) CP and dTCMV

Figure 4.2. *Illustration of the results of Theorem 4.13: CDFs of $W_j^{\mathcal{E}}(T)$, $j \in \{d, c, cd, cp\}$, all with the same expected value $\mathcal{E} = 250$. $w_0 = 100$, $t_0 = 0$, $T = 10$, $w_0 e^{rT} = 106.43$. Other parameters as in section 5.*



(a) cTCMV, CP, and PCMV (b) Same as figure (a) on the left, with left tail enlarged

Figure 4.3. *Illustration of the results of Theorem 4.13: CDFs of $W_j^{\mathcal{E}}(T)$, $j \in \{p, c, cp\}$, all with the same expected value $\mathcal{E} = 250$. $w_0 = 100$, $t_0 = 0$, $T = 10$, other parameters as in section 5. The value of $w^0_{cp;c}$ in (4.41)–(4.42) is indicated in both figures.*

Up to this point, we have only focused on the expectation $\mathcal{E}$ of the target terminal wealth distribution. However, the expectation conditional on $W_j^{\mathcal{E}}(T)$ being below the risk-free investment outcome $w_0 e^{rT}$ or simply conditional on underperforming the expectation target

$\mathcal{E}$ is also likely to be of particular interest to the investor. The following lemma summarizes the conditional expectation results for the investment strategies $j \in \{d, c, cd, cp\}$.

Lemma 4.14 (conditional expectations of target terminal wealth distributions). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied, and let $\phi(\cdot)$ and $\Phi(\cdot)$ be the PDF and CDF of the standard normal distribution, respectively. The conditional expectations of $W_j^{\mathcal{E}}(T)$, given that $W_j^{\mathcal{E}}(T) \leq w$, for $j \in \{d, c, cd, cp\}$, are as follows:*

(4.43)
$$E_{u_d^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_d^{\mathcal{E}}(T) \middle| W_d^{\mathcal{E}}(T) \leq w\right] = \mathcal{E} - \sqrt{\frac{(e^{AT}+1)}{2(e^{AT}-1)}} \cdot \left(\mathcal{E} - w_0 e^{rT}\right) \frac{\phi\left(\frac{(w-\mathcal{E})}{\mathcal{E}-w_0e^{rT}} \cdot \sqrt{\frac{2(e^{AT}-1)}{(e^{AT}+1)}}\right)}{\Phi\left(\frac{(w-\mathcal{E})}{\mathcal{E}-w_0e^{rT}} \cdot \sqrt{\frac{2(e^{AT}-1)}{(e^{AT}+1)}}\right)},$$

(4.44)
$$E_{u_c^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_c^{\mathcal{E}}(T) \middle| W_c^{\mathcal{E}}(T) \leq w\right] = \mathcal{E} - \frac{1}{\sqrt{AT}} \cdot \left(\mathcal{E} - w_0 e^{rT}\right) \frac{\phi\left(\frac{(w-\mathcal{E})}{\mathcal{E}-w_0e^{rT}} \cdot \sqrt{AT}\right)}{\Phi\left(\frac{(w-\mathcal{E})}{\mathcal{E}-w_0e^{rT}} \cdot \sqrt{AT}\right)},$$

(4.45)
$$E_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_j^{\mathcal{E}}(T) \middle| W_j^{\mathcal{E}}(T) \leq w\right] = \mathcal{E} \cdot \frac{\Phi\left(\frac{\log(w/\mathcal{E})-\frac{1}{2}\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2}{\hat{\sigma}_j^{\mathcal{E}}}\right)}{\Phi\left(\frac{\log(w/\mathcal{E})+\frac{1}{2}\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2}{\hat{\sigma}_j^{\mathcal{E}}}\right)}, \qquad j \in \{cd, cp\}.$$

Proof. This follows from Lemma 3.4 and Assumption 4.1. ∎

We now use the results of Lemma 4.14 to compare the expectations of the target terminal wealth distributions conditional on $W_j^{\mathcal{E}}(T) \leq w$, for any $w < \mathcal{E}$, where $j \in \{d, c, cd, cp\}$. The results, given in Lemma 4.15, are intuitively expected given the results up to this point.

Lemma 4.15 (Comparison: Conditional expectations for underperforming target $\mathcal{E}$). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. The conditional expected values of $W_j^{\mathcal{E}}(T)$, conditional on $W_j^{\mathcal{E}}(T) \leq w$, where $w < \mathcal{E}$ and $j \in \{d, c, cd, cp\}$, satisfy the following:*

$$E_{u_d^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_d^{\mathcal{E}}(T) \middle| W_d^{\mathcal{E}}(T) \leq w\right] < E_{u_c^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_c^{\mathcal{E}}(T) \middle| W_c^{\mathcal{E}}(T) \leq w\right] \quad \forall w < \mathcal{E}, \tag{4.46}$$

$$E_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cd}^{\mathcal{E}}(T) \middle| W_{cd}^{\mathcal{E}}(T) \leq w\right] \leq E_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_{cp}^{\mathcal{E}}(T) \middle| W_{cp}^{\mathcal{E}}(T) \leq w\right] \quad \forall w \in (0, \mathcal{E}). \tag{4.47}$$

Proof. The inverse Mills ratio $\lambda(x) := \phi(x)/\Phi(x)$ is strictly decreasing for all $x \in \mathbb{R}$, with $\lambda'(x) \in (-1, 0)$ for all x. Since $\lambda'(x) = -\lambda(x)[x + \lambda(x)]$ and $\lambda(x) > 0$ for all x, we have, in particular, $x + \lambda(x) > 0$ for all $x < 0$. Therefore, we have

$$\frac{d}{dx}\left[\frac{1}{x}\lambda(x)\right] < -\frac{1}{x^2}\left[x + \lambda(x)\right] < 0 \quad \forall x < 0, \tag{4.48}$$

so that the function $\frac{1}{x}\lambda(x)$ is strictly decreasing for all $x < 0$. Considering (4.43) and (4.44), together with the requirement that $w < \mathcal{E}$ and the inequality (4.16), this is sufficient to conclude (4.46). To prove (4.47), we fix some constant $c \geq 0$ and consider the auxiliary function $x \rightarrow f_\Phi(x; c)$ defined by

$$f_\Phi(x;c) = \frac{\Phi\left(-\frac{c}{x} - \frac{1}{2}x\right)}{\Phi\left(-\frac{c}{x} + \frac{1}{2}x\right)}, \qquad x \geq 0 \ \ (c \geq 0). \tag{4.49}$$

We observe that $f_\Phi \geq 0$, and $f'_\Phi(x;c) \leq 0$ if and only if

$$\left[\frac{c}{x^2} - \frac{1}{2}\right] \cdot \lambda\left(-\frac{c}{x} - \frac{1}{2}x\right) \leq \left[\frac{c}{x^2} + \frac{1}{2}\right] \cdot \lambda\left(-\frac{c}{x} + \frac{1}{2}x\right), \qquad x \geq 0 \ \ (c \geq 0). \tag{4.50}$$

If $\left[\frac{c}{x^2} - \frac{1}{2}\right] \leq 0$, then (4.50) holds since $\lambda(x)$ is positive and decreasing for all $x \in \mathbb{R}$. If $\left[\frac{c}{x^2} - \frac{1}{2}\right] > 0$ or, equivalently, $c > \frac{1}{2}x^2$, the inequality (4.50) also holds since $y \rightarrow \frac{1}{y}\lambda(y)$ for all $y < 0$ is decreasing as a result of (4.48). Therefore, since $f_\Phi(x;c)$ is decreasing in $x \geq 0$ for any fixed $c \geq 0$, the relationship (4.8) and expressions (4.45) imply the result (4.47). ■

The results of Lemma 4.15, while not making as general a statement as Theorem 4.13, are arguably of more practical relevance to investors since its conclusions are simple and intuitive to interpret. Informally, (4.46)–(4.47) simply state that when the investor is primarily concerned with outcomes underperforming the target $\mathcal{E}$, the DOMV and dTCMV strategies always lead to worse underperformance on average than the cTCMV and CP strategies, respectively.

Note that Lemma 4.15 does not also provide a comparison of the conditional expectations in the cases of CP and cTCMV. The reason is that such a comparison depends on the process and investment parameters in a fairly complicated way, and we instead explore the relationship between CP and cTCMV outcomes in more detail in the αVaR results below. Here we simply observe that since the cTCMV strategy can result in negative wealth outcomes, we do know that for some sufficiently small value[7] of $w_\delta > 0$ we have

$$E^{w_0,t_0=0}_{u_c^{\mathcal{E}*}}\left[W_c^{\mathcal{E}}(T) \middle| W_c^{\mathcal{E}}(T) \leq w\right] < E^{w_0,t_0=0}_{u_{cp}^{\mathcal{E}*}}\left[W_{cp}^{\mathcal{E}}(T) \middle| W_{cp}^{\mathcal{E}}(T) \leq w\right] \quad \text{for } w \in (0, w_\delta], \tag{4.51}$$

which turns out to be sufficient to explain the numerical results observed in section 5.

We introduce the following definition of the αVaR and αCVaR, which has been adapted from the definition given in Forsyth, Vetzal, and Westmacott (2019) to our setting. Note that, depending on the application, slightly different formulations are used in the literature (for example, focusing on the "loss distribution" instead; see Miller and Yang (2017); Rockafellar and Uryasev (2002)), but all these definitions have the same qualitative content.

Definition 4.16 (αVaR and αCVaR). *Fix a level $\alpha \in (0,1)$. The VaR at level α, or αVaR, is defined as the terminal wealth value $\alpha VaR^{w_0,t_0}_{u_j^*}$, where*

$$\alpha VaR^{w_0,t_0}_{u_j^*} := w_\alpha \ \ \text{such that} \ \ \alpha \equiv \mathbb{P}^{w_0,t_0}_{u_j^{\mathcal{E}*}}\left[W_j^{\mathcal{E}}(T) \leq w_\alpha\right], \qquad j \in \{p, d, c, cd, cp\}. \tag{4.52}$$

[7] The value of w_δ should be sufficiently small in the context of all the investment and process parameters. For example, in section 5 we give an example where $w_\delta > w_0 e^{rT}$.

The CVaR (also known as the expected shortfall) at level α, or αCVaR, is the expected value of terminal wealth $W_j^{\mathcal{E}}(T)$ given that it is below the level of the associated αVaR. In other words,

$$\alpha CVaR_{u_j^*}^{w_0,t_0} := E_{u_j^*}^{w_0,t_0}\left[W_j^{\mathcal{E}}(T)\middle| W_j^{\mathcal{E}}(T) \leq \alpha\, VaR_{u_j^*}^{w_0,t_0}\right], \qquad j \in \{p,d,c,cd,cp\}. \tag{4.53}$$

Note that according to Definition 4.16, all else being equal, *smaller* values of $\alpha\text{VaR}_{u_j^*}^{w_0,t_0}$ and $\alpha\text{CVaR}_{u_j^*}^{w_0,t_0}$ represent a *worse* outcome for the investor than larger values. This qualitative interpretation is of course the opposite of those examples in the literature, where these quantities are defined in terms of the loss distribution.

Typical values of α used in Definition 4.16 are fairly small, for example, $\alpha = 0.05$ (5%) or $\alpha = 0.01$ (1%). However, the following lemma compares the αVaR results for any choice of $\alpha \in (0, 0.5)$, since this interval is wide enough to ensure that all likely values of interest of α will be included.

Lemma 4.17 (Comparison: αVaR). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. Fix a level $\alpha \in (0, 0.5)$. The following comparison results hold for $\alpha\, VaR_{u_j^*}^{w_0,t_0=0}$, $j \in \{d,c,cd,cp\}$:*

$$\alpha\, VaR_{u_d^{\mathcal{E}*}}^{w_0,t_0=0} < \alpha\, VaR_{u_c^{\mathcal{E}*}}^{w_0,t_0=0} \quad \forall \alpha \in (0,0.5), \tag{4.54}$$

$$\alpha\, VaR_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0} \leq \alpha\, VaR_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0} \quad \forall \alpha \in (0,0.5). \tag{4.55}$$

Proof. This follows from the results of Theorem 4.13. However, a direct proof is instructive due to the key role played by αVaR in the risk management literature (Jorion (2009)). We start by noting that the definition (4.52), together with the results of Lemma 4.9, implies that

$$\alpha\text{VaR}_{u_d^{\mathcal{E}*}}^{w_0,t_0=0} = \mathcal{E} + \sqrt{\frac{(e^{AT}+1)}{2\,(e^{AT}-1)}}\left(\mathcal{E} - w_0 e^{rT}\right)\cdot \Phi^{-1}(\alpha), \tag{4.56}$$

$$\alpha\text{VaR}_{u_c^{\mathcal{E}*}}^{w_0,t_0=0} = \mathcal{E} + \frac{1}{\sqrt{AT}}\left(\mathcal{E} - w_0 e^{rT}\right)\cdot \Phi^{-1}(\alpha), \tag{4.57}$$

$$\alpha\text{VaR}_{u_j^{\mathcal{E}*}}^{w_0,t_0=0} = \mathcal{E}\cdot \exp\left\{\hat{\sigma}_j^{\mathcal{E}}\cdot \Phi^{-1}(\alpha) - \frac{1}{2}\left(\hat{\sigma}_j^{\mathcal{E}}\right)^2\right\}, \quad j \in \{cd,cp\}. \tag{4.58}$$

The result (4.54) therefore follows from (4.56)–(4.57) and the inequality (4.16), together with the fact that $\Phi^{-1}(\alpha) < 0$ for all $\alpha < 0.5$. Next, we observe that if $\hat{\sigma}_{cp}^{\mathcal{E}} = \hat{\sigma}_{cd}^{\mathcal{E}}$, then it is clear that $\alpha\text{VaR}_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0} = \alpha\text{VaR}_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}$. Assume therefore that $\hat{\sigma}_{cp}^{\mathcal{E}} < \hat{\sigma}_{cd}^{\mathcal{E}}$. Then (4.58) implies that $\alpha\text{VaR}_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0} < \alpha\text{VaR}_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}$ for all $\alpha > 0$ such that $\alpha < \Phi\left(\frac{1}{2}\left[\hat{\sigma}_{cd}^{\mathcal{E}} + \hat{\sigma}_{cp}^{\mathcal{E}}\right]\right)$. Observing that $0.5 < \Phi\left(\frac{1}{2}\left[\hat{\sigma}_{cd}^{\mathcal{E}} + \hat{\sigma}_{cp}^{\mathcal{E}}\right]\right)$, we see that the result (4.55) also holds. ∎

Given the results of Theorem 4.13 and Lemma 4.17, as well as the fact that the αVaR might be of particular interest to investors, we analyze the αVaR results for the CP and cTCMV strategies in more detail. To this end, we give the following simple initial result.

Lemma 4.18 (Comparison: αVaR for CP and cTCMV, a simple condition). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. Then*

$$\alpha\, VaR_{u_c^{\mathcal{E}*}}^{w_0,t_0=0} < \alpha\, VaR_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0} \quad \text{if } \alpha < \Phi\left(-\frac{\mathcal{E}}{(\mathcal{E}-w_0e^{rT})}\cdot\sqrt{AT}\right). \tag{4.59}$$

Proof. By Lemma 3.4, $W_c^{\mathcal{E}}(T)$ can assume negative values but $W_{cp}^{\mathcal{E}}(T)$ cannot. Therefore, if α is chosen such that $\alpha\mathrm{VaR}_{u_c^*}^{w_0,t_0=0} < 0$, then it necessarily follows that $\alpha\mathrm{VaR}_{u_c^*}^{w_0,t_0=0} < \alpha\mathrm{VaR}_{u_{cp}^*}^{w_0,t_0=0}$. The condition on α in (4.59) follows from the expression for $\alpha\mathrm{VaR}_{u_c^{\mathcal{E}*}}^{w_0,t_0=0}$ in (4.57), ensuring that $\alpha\mathrm{VaR}_{u_c^*}^{w_0,t_0=0} < 0$. ■

The result of Lemma 4.18 is useful in that it is easy to verify, and if α is small, the condition (4.59) is often easily satisfied; for example, it is sufficient to explain the 1%VaR results for CP and cTCMV reported in section 5. However, if we consider more general values for $\alpha \in (0, 0.5)$, the comparison results of αVaR for CP and cTCMV are more involved, as the following lemma shows. Specifically, we give two conditions on the process and investment parameters, either of which can be used to obtain more specific comparison results regarding αVaR for CP and cTCMV.

Lemma 4.19 (Comparison: αVaR for CP and cTCMV). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. Furthermore, assume that the wealth process (2.1) and investment parameters are such that either condition $C1$ or condition $C2$ is satisfied, where*

(4.60)
$$C1:\log^2\left(\frac{\mathcal{E}}{w_0e^{rT}}\right)\cdot\exp\left\{-\frac{1}{2AT}\log^2\left(\frac{\mathcal{E}}{w_0e^{rT}}\right)\right\} > \frac{2}{5}\sqrt{AT}\left(\frac{\mathcal{E}-w_0e^{rT}}{\mathcal{E}}\right),$$

(4.61)
$$C2:\frac{1}{\sqrt{A\;T}}\left[\log\left(\frac{\mathcal{E}}{w_0}\right)-rT\right]^2\cdot\exp\left\{-\frac{1}{2AT}\left[\log\left(\frac{\mathcal{E}}{w_0}\right)-rT\right]^2\right\} > \frac{2}{5}\frac{(\mathcal{E}-w_0e^{rT})}{\mathcal{E}}.$$

Then there exists a unique value $\alpha_{cp;c} \in (0, 0.5)$ such that

$$\alpha\, VaR_{u_c^{\mathcal{E}*}}^{w_0,t_0=0} < \alpha\, VaR_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0} \quad \forall\alpha\in(0,\alpha_{cp;c}), \tag{4.62}$$

$$\alpha\, VaR_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0} < \alpha\, VaR_{u_c^{\mathcal{E}*}}^{w_0,t_0=0} \quad \forall\alpha\in(\alpha_{cp;c},0.5), \tag{4.63}$$

while the difference $[\alpha\, VaR_{u_c^{\mathcal{E}*}}^{w_0,t_0=0} - \alpha\, VaR_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}]$ attains a maximum at $\alpha^* \in (\alpha_{cp;c}, 1)$ given by

$$\alpha^* = \Phi\left(\frac{\sqrt{AT}}{\log(\mathcal{E}/w_0)-rT}\cdot\log\left(\frac{1-\frac{\mathcal{E}}{w_0e^{rT}}}{\log(\mathcal{E}/w_0)-rT}\right)+\frac{1}{2}\cdot\frac{\log(\mathcal{E}/w_0)-rT}{\sqrt{AT}}\right). \tag{4.64}$$

Proof. From Lemma 4.18, we know that $\alpha\mathrm{VaR}_{u_c^*}^{w_0,t_0=0} < \alpha\mathrm{VaR}_{u_{cp}^*}^{w_0,t_0=0}$ provided α is sufficiently small. From the results (4.57)–(4.58), it is clear that $\alpha\mathrm{VaR}_{u_{cp}^*}^{w_0,t_0=0} < \alpha\mathrm{VaR}_{u_c^*}^{w_0,t_0=0}$ if $\alpha = 0.5$

and, by continuity, therefore also for some ϵ-neighborhood of $\alpha = 0.5$. It is straightforward to show that either of the relatively simple conditions (4.60)–(4.61) is sufficient to ensure that the function $\alpha \to [\alpha\mathrm{VaR}_{u_c^{\mathcal{E}*}}^{w_0,t_0=0} - \alpha\mathrm{VaR}_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0}]$ is strictly concave, so that the results (4.62)–(4.64) follow. ■

The results of Lemma 4.19 are useful in providing an explanation of the numerical results presented in section 5, where we encounter a particular example where both conditions (4.60) and (4.61) are satisfied and $\alpha_{cp;c} \in (0.05, 0.1)$.

Given the recent interest in using αCVaR as a risk measure in dynamic portfolio optimization applications (see, for example, Forsyth (2020); Miller and Yang (2017)), the following lemma compares the αCVaR results for investment strategies $j \in \{d, c, cd, cp\}$ for any choice $\alpha \in (0, 1)$. We highlight the fact that while the conditional expectation comparison (Lemma 4.15) compares the results below a fixed wealth level regardless of the associated percentile, the αCVaR comparison in Lemma 4.20 considers the conditional expectations of wealth outcomes below a fixed percentile (see Definition 4.16).

Lemma 4.20 (Comparison: αCVaR). *Assume that the conditions of Assumptions 3.1 and 4.1 are satisfied. Fix a level $\alpha \in (0, 1)$. The following comparison results hold for $\alpha CVaR_{u_j^*}^{w_0,t_0=0}$, $j \in \{d, c, cd, cp\}$:*

$$\alpha CVaR_{u_d^{\mathcal{E}*}}^{w_0,t_0=0} < \alpha CVaR_{u_c^{\mathcal{E}*}}^{w_0,t_0=0} \quad \forall \alpha \in (0, 1), \tag{4.65}$$

$$\alpha CVaR_{u_{cd}^{\mathcal{E}*}}^{w_0,t_0=0} \le \alpha CVaR_{u_{cp}^{\mathcal{E}*}}^{w_0,t_0=0} \quad \forall \alpha \in (0, 1). \tag{4.66}$$

Proof. Given Definition 4.16, the results of Lemma 4.15, and the results for $\alpha\mathrm{VaR}_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}$ in (4.56)–(4.58), we have the following expressions for $\alpha\mathrm{CVaR}_{u_j^*}^{w_0,t_0=0}$, $j \in \{d, c, cd, cp\}$:

$$\alpha\mathrm{CVaR}_{u_d^{\mathcal{E}*}}^{w_0,t_0=0} = \mathcal{E} - \sqrt{\frac{(e^{AT}+1)}{2\,(e^{AT}-1)}} \cdot \left(\mathcal{E} - w_0 e^{rT}\right) \cdot \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{\alpha}, \tag{4.67}$$

$$\alpha\mathrm{CVaR}_{u_c^{\mathcal{E}*}}^{w_0,t_0=0} = \mathcal{E} - \frac{1}{\sqrt{AT}}\left(\mathcal{E} - w_0 e^{rT}\right) \cdot \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{\alpha}, \tag{4.68}$$

$$\alpha\mathrm{CVaR}_{u_j^{\mathcal{E}*}}^{w_0,t_0=0} = \mathcal{E} \cdot \frac{\Phi\left(\Phi^{-1}(\alpha) - \hat{\sigma}_j^{\mathcal{E}}\right)}{\alpha}, \qquad j \in \{cd, cp\}. \tag{4.69}$$

Since $\phi(x) > 0$ for all x and $\alpha > 0$, the result (4.65) follows from the inequality (4.16) together with (4.67)–(4.68). Second, (4.66) follows from (4.69) together with (4.8) and the fact that Φ is strictly increasing. ■

The results of Lemma 4.20 are intuitively expected given the results of Lemmas 4.15 and 4.17. We do not provide a comparison of αCVaR in the cases of CP and cTCMV, since such a comparison is too cumbersome to be of much practical use—this can be seen by comparing the requirement of Definition 4.16 with the αVaR results in Lemma 4.19.

In the next section, we present numerical results illustrating the analytical results presented in this section.

5. Numerical results. To obtain the numerical results presented in this section, we assume a fixed initial wealth of $w_0 = 100$ at time $t_0 \equiv 0$, and an investment time horizon of $T = 10$ years. The wealth dynamics (2.1) is parameterized using the same calibration data and calibration techniques as detailed in Dang and Forsyth (2016); Forsyth and Vetzal (2017a), which we now briefly summarize. In terms of the empirical data sources, the risky asset data are based on inflation-adjusted daily total return data (including dividends and other distributions) for the period 1926–2014 from the CRSP's VWD index,[8] which is a capitalization-weighted index of all domestic stocks on major US exchanges. The risk-free rate is based on 3-month US T-bill rates[9] over the period 1934–2014 and has been augmented with the NBER's short-term government bond yield data[10] for 1926–1933 to incorporate the impact of the 1929 stock market crash. Prior to calculations, all time series were inflation-adjusted using data from the US Bureau of Labor Statistics.[11] Standard maximum likelihood techniques are used to calibrate the GBM dynamics; see Dang and Forsyth (2016); Forsyth and Vetzal (2017a) for more information regarding the relevant details. As a result, we obtain the following parameters for use throughout this section:

$$\mu = 0.0816, \qquad \sigma = 0.1863, \qquad r = 0.00623. \tag{5.1}$$

Table 5.1 presents the numerical results on various aspects of the target terminal wealth distributions for two expected value targets, $\mathcal{E} = 125$ and $\mathcal{E} = 250$. Note that investing all wealth in the risk-free asset over the entire time period $[0, T]$ results in a terminal wealth of $w_0 e^{rT} = 106.43$. Therefore, the strategies associated with the target $\mathcal{E} = 125$ are quite risk-averse but not to the extent that all wealth is invested in the risk-free asset. In contrast, a target of $\mathcal{E} = 250$ requires a substantial investment in the risky asset during at least a significant portion of the investment time period.

We make the following observations regarding the results in Table 5.1:

- The role of the expected value target in shaping the results is worth highlighting. Specifically, the larger the expected value target, the larger the investment required in the risky asset, which magnifies the differences between the investment strategies, as expected. As a result, for purposes of clarity we focus mostly on the results for the target $\mathcal{E} = 250$ in the subsequent discussion.
- The first-order stochastic dominance results of Theorem 4.13 are illustrated quite dramatically in Table 5.1. It is clear from the results that, subject to the stated assumptions under which these results were derived, no rational investor purely interested in the terminal wealth distributions would pursue the DOMV-optimal or the dTCMV-optimal strategy, since the cTCMV-optimal and CP strategies, respectively, perform much better.

[8]Calculated based on data from the Historical Indexes 2015©, Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third party suppliers.

[9]Data was obtained from http://research.stlouisfed.org/fred2/series/TB3MS.

[10]Data was obtained from the National Bureau of Economic Research (NBER) website, http://www.nber.org/databases/macrohistory/contents/chapter13.html.

[11]The annual average CPI-U index, which is based on inflation data for urban consumers, was used; see https://www.bls.gov/cpi.

- The performance of the dTCMV-optimal strategy can be exceptionally poor. Of course, while this has been established convincingly by the results presented in section 4, the sheer degree of the underperformance can be quite dramatic, as the case of $\mathcal{E} = 250$ highlights. Observe, for example, that in this case the standard deviation of $W_{cd}^{\mathcal{E}}(T)$ is more than two times that of $W_{cp}^{\mathcal{E}}(T)$, about four times that of $W_{c}^{\mathcal{E}}(T)$, and more than *six times* that of $W_{p}^{\mathcal{E}}(T)$. The median of $W_{cd}^{\mathcal{E}}(T)$ is also exceptionally poor, and there is a 45% chance that $W_{cd}^{\mathcal{E}}(T)$ is below $w_0 e^{rT}$. Arguably the only redeeming feature of $W_{cd}^{\mathcal{E}}(T)$ is the role of its log-normal distribution in limiting the downside tail risk in the most extreme cases; this is illustrated by the 1%VaR and 1%CVaR results. However, the same can be said of the corresponding CP strategy, which per Theorem 4.13 performs much better overall than the dTCMV strategy. Since the poor performance of the dTCMV strategy has also been confirmed in Forsyth and Vetzal (2019a) using numerical experiments for the case where multiple realistic investment constraints are applied simultaneously, there is some concern about the popularity of applying the dTCMV approach in institutional settings in the literature (see, for example, Bi and Cai (2019); Li and Li (2013), Liang, Bai, and Guo (2014); Sun, Li, and Zeng (2016); Wang and Chen (2018, 2019); Long and Zeng (2016); Peng, Cui, and Shi (2018); Zhang et al. (2017)).
- The cTCMV-optimal strategy performs very well compared to the CP strategy by a number of the measures considered, for example, standard deviation and the probability that the terminal wealth will fall below $w_0 e^{rT}$ or the target $\mathcal{E}$. However, the CP strategy performs better where the extreme left tail of the distribution is concerned (for example, the αVaR and αCVaR for $\alpha \in \{1\%, 5\%\}$), which agrees with the numerical results presented in Forsyth and Vetzal (2019a) and also confirms the analytical conclusions of section 4, especially Theorem 4.13.
- The PCMV-optimal strategy has the best performance in terms of the standard deviation (Lemma 4.4) and also in terms of the median wealth (Lemma 4.8). However, as observed in Forsyth and Vetzal (2019a), this performance comes at the cost of increased left tail risk, as confirmed by our negative skewness and excess kurtosis results for the distribution of $W_{p}^{\mathcal{E}}(T)$; see Lemmas 4.6 and 4.7. The implication in this example is that the resulting 1%VaR and 1%CVaR are the worst of all the strategies considered. However, this is only true for very extreme tail outcomes, since already the 5%VaR and 5%CVaR associated with $W_{p}^{\mathcal{E}}(T)$ are the best of all the strategies considered.

Finally, we note that while the numerical results presented in Table 5.1 illustrate the analytical results of section 4, and are therefore also subject to Assumptions 3.1 and 4.1, the qualitative observations regarding the relative performance of the different strategies are in agreement with the observations from the relevant numerical results available in the literature. In particular, we refer the reader to Forsyth and Vetzal (2017b, 2019a,b) and Forsyth, Vetzal, and Westmacott (2019), where the portfolio optimization problems are solved numerically subject to multiple realistic investment constraints being applied simultaneously. This illustrates that our analytical results, while obtained under stylized assumptions regarding trading in the underlying market, are nevertheless of practical use in explaining the performance of dynamic MV-optimal investment strategies in a realistic setting.

Table 5.1

Numerical results related to the target terminal wealth distributions for two expected value targets, $\mathcal{E} = 125$ and $\mathcal{E} = 250$. Initial wealth $w_0 = 100$, $t_0 = 0$, and $T = 10$ years. "Parameter" reports the values of $\gamma_p^{\mathcal{E}}$, $\rho_d^{\mathcal{E}}$, $\rho_c^{\mathcal{E}}$, $\rho_{cd}^{\mathcal{E}}/(2w_0)$, and $\theta_{cp}^{\mathcal{E}}$, respectively, for each strategy achieving the stated expected value target $\mathcal{E}$ as per (4.2)–(4.6). "Prob.$\leq k$" refers to the probability $\mathbb{P}_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_j^{\mathcal{E}}(T) \leq k\right]$, and "CExp.$\leq k$" to the conditional expectation $E_{u_j^{\mathcal{E}*}}^{w_0,t_0=0}\left[W_j^{\mathcal{E}}(T) \middle| W_j^{\mathcal{E}}(T) \leq k\right]$, respectively, for $j \in \{p, d, c, cd, cp\}$. Numbers rounded to nearest integer except where doing so would obscure relevant information.

Quantity	Target expected value $\mathcal{E} = 125$					Target expected value $\mathcal{E} = 250$				
	PCMV	DOMV	cTCMV	dTCMV	CP	PCMV	DOMV	cTCMV	dTCMV	CP
Parameter	259	0.111	0.044	0.041	0.213	569	0.014	0.006	0.001	1.133
Mean	125	125	125	125	125	250	250	250	250	250
Median	127	125	125	124	124	269	250	250	123	200
Stdev	9	16	15	16	16	71	124	112	444	187
Skewness	-15	0	0	0.4	0.4	-15	0	0	11	3
Ex.Kurtosis	1042	0	0	0.3	0.3	1042	0	0	487	15
1% VaR	91	88	91	92	93	-15	-38	-11	8	42
5% VaR	113	99	101	101	101	159	47	65	17	67
10% VaR	119	105	106	105	106	206	92	106	27	85
1% CVaR	63	82	86	89	89	-228	-80	-49	5	34
5% CVaR	97	92	95	96	96	37	-5	19	11	52
10% CVaR	107	97	100	99	100	112	33	53	17	64
Prob.$\leq w_0 e^{rT}$	3%	12%	10%	11%	11%	3%	12%	10%	45%	17%
Prob.$\leq \mathcal{E}$	26%	50%	50%	53%	53%	26%	50%	50%	72%	63%
CExp.$\leq w_0 e^{rT}$	87	99	100	100	100	-45	45	53	52	77
CExp. $\leq \mathcal{E}$	117	112	113	113	113	187	151	160	95	146

6. Conclusion.

In this paper, we compared the terminal wealth distributions obtained by implementing the optimal investment strategies associated with the different approaches to dynamic MV optimization available in the literature. In particular, we considered the precommitment MV (PCMV) approach, the dynamically optimal MV (DOMV) approach, and the time-consistent MV approach with a constant risk-aversion parameter (cTCMV) and wealth-dependent risk-aversion parameter (dTCMV), respectively. For comparison and benchmarking purposes, a constant proportion (CP) strategy was also considered.

We introduced some simplifying assumptions regarding the underlying market in order to analytically compare the resulting terminal wealth distributions on a fair basis. Specifically, we assumed that the investor is agnostic about the philosophical differences underlying the various approaches to MV optimization and simply wishes to achieve a chosen expected value of terminal wealth regardless of the approach. We also assumed that the investor faced no leverage constraints or transaction costs and could trade continuously in the market.

Subject to these assumptions, we presented first-order stochastic dominance results proving that for wealth outcomes below the chosen expected value target, the cTCMV strategy always outperforms the DOMV strategy, and the CP strategy always outperforms the dTCMV strategy. We also showed that the dTCMV strategy performs exceptionally poorly among the strategies considered according to a number of criteria, including variance and median of terminal wealth, raising concerns regarding the popularity in the literature of applying the dTCMV strategy in institutional settings. Furthermore, we showed that the PCMV-optimal terminal wealth distribution has fundamentally different characteristics than any of the other strategies, including some characteristics which may be desirable (higher median, lower standard deviation) and some which may be less desirable (large negative skewness and excess kurtosis).

Our analytical results, while derived under simplifying assumptions, nonetheless prove effective in explaining the numerical results incorporating realistic investment constraints currently available in the literature

Finally, we leave further analysis of the PCMV-optimal target terminal wealth distribution, extension of our results to solutions for multiple risky assets, and treatment of alternative model specifications (e.g., jumps in the risky asset process and alternative model specifications) for our future work.

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